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A Note on Operator Equations Describing the Integral

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We study operator equations generalizing the chain rule and the substitution rule for the integral and the derivative of the type

$$f \circ g + c = I\ (Tf \circ g \cdot Tg), \quad f, g \in C^1(\mathbb{R}), \tag{1}$$

where $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ and where I is defined on $C(\mathbb{R})$. We consider suitable conditions on I and T such that (1) is well-defined and, after reformulating (1) as

$$V(f \circ g) = Tf \circ g \cdot Tg, \quad f, g \in C^1(\mathbb{R}) \tag{2}$$

with $V : C^1(\mathbb{R}) \to C(\mathbb{R})$, give the general form of T, V and I. Simple initial conditions then guarantee that the derivative and the integral are the only solutions for T and I. We also consider an analogue of the Leibniz rule and study surjectivity properties there.

Key words: operator equation, chain rule, Leibniz rule, integral.

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This note is dedicated to the memory of the famous expert in Geometric Functional Analysis M.I. Kadets. The second named author has blessed memories of his personal contacts with the great personality of Mishail Iosifovich Kadets, and considers himself fortunate that he has had this opportunity.

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1. Introduction and Preliminary Discussion

Generalizing the chain rule $D(f \circ g) = Df \circ g \cdot Dg$ for $f, g \in C^1(\mathbb{R})$, we studied in [AKM] the operator equation $T(f \circ g) = Tf \circ g \cdot Tg$ for non-degenerate operators $T : C^1(\mathbb{R}) \to C(\mathbb{R})$. These operators turned out to be of the form

$$Tf(x) = \frac{H \circ f(x)}{H(x)} \, |f'(x)|^p \{\operatorname{sgn} f'(x)\}$$

for a suitable function $H \in C(\mathbb{R})$, $H > 0$, a number $p \geq 0$ and where the term $\{\operatorname{sgn} f'(x)\}$ may be present or not, cf. Theorem 1 of [AKM]. The more general equation $V(f \circ g) = T_1 f \circ g \cdot T_2 g$; $f, g \in C^1(\mathbb{R})$ for operators $V, T_1, T_2 : C^1(\mathbb{R}) \to C(\mathbb{R})$ has, up to multiplication by continuous functions, very similar solutions, cf. Theorem 3 of [KM2].

Looking at the indefinite integral J and the derivative D, the chain rule takes the form

$$f \circ g + c = J \ (Df \circ g \cdot Dg), \quad f, g \in C^1(\mathbb{R}),$$

with c being a constant. Motivated by this equation, we look for operators $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ and I defined on $C(\mathbb{R})$ such that

$$f \circ g \sim I \ (Tf \circ g \cdot Tg) \tag{1}$$

holds for all $f, g \in C^1(\mathbb{R})$. The equivalence $\sim$ has to be understood in a way so that (1) yields a well-defined operator I: Assume, e.g., that $\sim$ means equality. Then, choosing $g = c$ to be a constant function yields for all $f \in C^1(\mathbb{R})$

$$f(c) = I((Tf)(c) \cdot Tc). \tag{3}$$

Assuming that T is non-degenerate in the sense that for any $c \in \mathbb{R}$ there are functions $f_1, f_2 \in C^1(\mathbb{R})$ with $f_1(c) = f_2(c)$ and $Tf_1(c) \neq Tf_2(c)$, we find that

$$I(Tf_1(c) \cdot Tc) = f_1(c) = f_2(c) = I(Tf_2(c) \cdot Tc),$$

so that either $Tc = 0$ holds for all constant functions or I will not be injective. If $Tc = 0$, using (1) for $f = c$ and general g and (3) for general $f \in C^1(\mathbb{R})$, we arrive at the conclusion

$$c = I((Tc) \circ g \cdot Tg) = I(0) = I((Tf)(c) \cdot Tc) = f(c).$$

Therefore, if $Tc = 0$, as in the case of the derivative and the indefinite integral, the image of I should consist of classes of functions modulo the constants. Let $\mathcal{C} \subset C^1(\mathbb{R})$ denote the constant functions. To make (1) a meaningful equation (and also motivated by the indefinite integral) we may require that there are maps

(a) $I : C(\mathbb{R}) \to C^1(\mathbb{R})/\mathcal{C}$ and $T : C^1(\mathbb{R})/\mathcal{C} \to C(\mathbb{R})$ satisfying (1) with I being injective.

For $f \in C^1(\mathbb{R})$, denote $[f] := f + \mathcal{C} \in C^1(\mathbb{R})/\mathcal{C}$. Equation (1) then might be interpreted as

$$[f \circ g] = I \ (T[f] \circ g \cdot T[g]) \quad ; \ f, g \in C^1(\mathbb{R}). \tag{1'}$$

Note here that $[f] \circ g = [f \circ g]$.

Alternatively, motivated by the definite integral, we may ask that there are operators

(b) $I : C(\mathbb{R}) \to C^1(\mathbb{R})$ and $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ and a fixed number $c \in \mathbb{R}$ such that I is injective and

$$f \circ g - (f \circ g)(c) = I(Tf \circ g \cdot Tg); \quad f, g \in C^1(\mathbb{R}) \tag{1}$$

holds. In the next section we give precise statements describing the solutions of the operator equations (1') and (1).

2. Results for the Chain Rule

To state the results, we need the following notion of non-degeneracy of T.

Definition 1. *A map $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ is called* non-degenerate *provided that there is $y \in \mathbb{R}$ such that for any $x \in \mathbb{R}$ there is $f \in C_b^1(\mathbb{R})$ with $f(x) = y$ and $(Tf)(x) \neq 0$. Here $C_b^1(\mathbb{R})$ denotes the half-bounded C^1-functions on $\mathbb{R}$, i.e., bounded from above or below (or both). We use a corresponding definition if T acts as $T : C^1(\mathbb{R})/\mathcal{C} \to C(\mathbb{R})$.*

In case (b) we have the following result.

Theorem 1. *Assume that $I : C(\mathbb{R}) \to C^1(\mathbb{R})$ and $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ are operators such that for some fixed $c \in \mathbb{R}$*

$$f \circ g - (f \circ g)(c) = I(Tf \circ g \cdot Tg), \quad f, g \in C^1(\mathbb{R}) \tag{1}$$

holds. Suppose further that T is non-degenerate and that I is injective. Then there are constants $p > 0$, $d \neq 0$ such that

$$\begin{aligned} Tf(x) &= d \ |f'(x)|^p \ (\operatorname{sgn} f'(x)), \quad f \in C^1(\mathbb{R}), \\ Ih(x) &= d^{-2/p} \int_c^x |h(s)|^{1/p} \operatorname{sgn} h(s) \, ds, \quad h \in C(\mathbb{R}). \end{aligned} \tag{4}$$

If T satisfies the initial conditions $T(2\mathrm{Id}) = 2$ and $T(3\mathrm{Id}) = 3$ (the constant functions 2 and 3), we have that $p = 1$ and $d = 1$,

$$Tf(x) = f'(x), \quad Ih(x) = \int_c^x h(s)ds.$$

Hence T is a generalized derivative and I a generalized definite integral. The two initial conditions may be replaced by $T(b\mathrm{Id}) = b$ for two different constants $b \in \mathbb{R}$ different from 0 and 1. Case (a) leads to an analogue of the indefinite integral.

Theorem 2. *Assume that $I : C(\mathbb{R}) \to C^1(\mathbb{R})/\mathcal{C}$ and $T : C^1(\mathbb{R})/\mathcal{C} \to C(\mathbb{R})$ are operators such that*

$$[f \circ g] = I\ (T[f] \circ g \cdot T[g]), \quad f, g \in C^1(\mathbb{R}) \tag{1'}$$

holds. Suppose further that T is non-degenerate and that there is $W : C^1(\mathbb{R}) \to C(\mathbb{R})$ such that $WI : C(\mathbb{R}) \to C(\mathbb{R})$ is injective. Then there are constants $p > 0$, $d \neq 0$ and such that

$$\begin{aligned} T[f](x) &= d\ |f'(x)|^p\ (\operatorname{sgn} f'(x)), \quad f \in C^1(\mathbb{R}), \\ Ih(x) &= d^{-2/p} \int^x |h(s)|^{1/p} \operatorname{sgn} h(s)\, ds + \mathcal{C},\ h \in C(\mathbb{R})\ . \end{aligned}$$

P r o o f of Theorem 1. Put $d := T(Id)$, $d \in C(\mathbb{R})$. Choose $g = \mathrm{Id}$ in (1) to find that $f - f(c) = I(d \cdot Tf)$, where $f(c)$ denotes the constant function with value $f(c)$. Since this holds for all $f \in C^1(\mathbb{R})$, d cannot be identically zero and I is surjective onto the space $C^1_c(\mathbb{R}) := \{h \in C^1(\mathbb{R}) |\, h(c) = 0\}$ of C^1-functions which are zero in c. Since I is injective by assumption, I is bijective as a map $I : C(\mathbb{R}) \to C^1_c(\mathbb{R})$. Denote its inverse by $\tilde{V}$, $\tilde{V} : C^1_c(\mathbb{R}) \to C(\mathbb{R})$ and define $V : C^1(\mathbb{R}) \to C(\mathbb{R})$ by $Vf := \tilde{V}(f - f(c))$. Applying I^{-1} to (1) yields

$$V(f \circ g) = \tilde{V}(f \circ g - (f \circ g)(c)) = Tf \circ g \cdot Tg \tag{2}$$

for all $f, g \in C^1(\mathbb{R})$. Choosing $g = Id$ and $f = Id$, respectively, we find that $Vf = d\, Tf$ and $Vg = (d \circ g)\, Tg$, i.e. for any $f \in C^1(\mathbb{R})$,

$$Vf = d \circ f \cdot Tf = d \cdot Tf.$$

Since T is assumed to be non-degenerate, there is $y \in \mathbb{R}$ such that for any $x \in \mathbb{R}$ there is $f \in C^1(\mathbb{R})$ with $f(x) = y$ and $Tf(x) \neq 0$. By the preceeding equality, $d(y)(Tf)(x) = d(x)(Tf)(x)$, i.e., d is a constant function with constant $d \neq 0$

since d was not identically zero. Define $Sf := Tf/d$. Then $Vf = d^2Sf, Tf = dSf$ and by (2)

$$S(f \circ g) = Sf \circ g \cdot Sg, \quad f, g \in C^1(\mathbb{R})$$

holds. Since T is non-degenerate and $d \neq 0$, also S is non-degenerate. Hence by Theorem 1 of [AKM] there is $H \in C(\mathbb{R})$, $H > 0$ and $p \geq 0$ such that either

$$Sf(x) = \frac{H \circ f}{H}|f'|^p, \quad p \geq 0,\ f \in C^1(\mathbb{R}),$$

or

$$Sf(x) = \frac{H \circ f}{H}|f'|^p \operatorname{sgn} f', \quad p > 0,\ f \in C^1(\mathbb{R}).$$

We indicate by brackets $\{\quad\}$ that the term $\operatorname{sgn} f'$ may be present or not in the solution formulas. Then the operators V, T satisfying (2) with T being non-degenerate are of the form

$$Vf = d^2\ \frac{H \circ f}{H}|f'|^p\{\operatorname{sgn} f'\},\ Tf = d\ \frac{H \circ f}{H}|f'|^p\{\operatorname{sgn} f'\} \tag{5}$$

with $H > 0$, $d \neq 0$ and $p \geq 0$.

Let $b \in \mathbb{R}$. Applying (1) to $g = \mathrm{Id}$ and f as well as $f + b$ yields

$$f - f(c) = I(d \cdot T(f + b)) = I(d \cdot Tf).$$

The injectivity of I together with $d \neq 0$ implies that $T(f + b) = Tf$, i.e., T does not depend on shifts by b. Therefore (5) yields for $f = \mathrm{Id}$ that $H(x + b) = H(x)$ for all $x \in \mathbb{R}$ which means that H is constant. Therefore $Tf = d\ |f'|^p\{\operatorname{sgn} f'\}$, and choosing $g = \mathrm{Id}$ in (1) we have

$$f - f(c) = I(d\ Tf) = I(d^2\ |f'|^p\ \{\operatorname{sgn} f'\}) =: Ih.$$

Since $I : C(\mathbb{R}) \to C_c^1(\mathbb{R})$ is bijective and defined also on all negative functions, the $\operatorname{sgn} f'$-term has to be present in the right side and $p > 0$ is required. To find a formula for I, we have to solve $h = d^2\ |f'|^p \operatorname{sgn}(f')$, i.e., $f' = d^{-2/p}\ |h|^{1/p} \operatorname{sgn}(h)$. Since $Ih(c) = 0$ is required, this gives that

$$Ih(x) = f(x) - f(c) = d^{-2/p} \int_c^x\ |h(s)|^{1/p} \operatorname{sgn} h(s)\, ds.$$

Clearly these operators satisfy Eq. (1). In the case that additionally $T(2\mathrm{Id}) = 2$ and $T(3\mathrm{Id}) = 3$, we have $p = 1, d = 1$. ■

P r o o f of Theorem 2. Choosing $g = Id$ in (1') shows that I is surjective onto $C^1(\mathbb{R})/\mathcal{C}$. Let $V := I^{-1} :\ C^1(\mathbb{R})/\mathcal{C} \to C(\mathbb{R})$. Then

$$V([f \circ g]) = T[f] \circ g \cdot T[g]; \quad f, g \in C^1(\mathbb{R}) \tag{2'}$$

holds. This is similar as in (2), however, here T and V are defined on function classes only. Equation (2') has similar solutions as (2) in terms of H, p and f', cf. (5). The requirement that $T[f]$ depends only on the class $[f] = f + \mathcal{C}$ again implies that H is constant, being invariant under shifts by constants b. Then with d, p as before

$$V[f] = d^2 \; |f'(x)|^p \; \{\operatorname{sgn} f'\}, \; T[f] = d \; |f'(x)|^p \; \{\operatorname{sgn} f'\},$$

$V[f] = I^{-1}[f], [f] = Ih$. Again we solve

$$h = V[f] = d^2 \; |f'|^p \; \{\operatorname{sgn} f'\} \tag{6}$$

also for non-positive functions h requires the term $\operatorname{sgn} f'$ to be present in V and T. We have

$$f' = d^{-2/p} \; |h|^{1/p} \; \operatorname{sgn} h$$

and hence

$$[f](x) = d^{-2/p} \int^x |h(s)|^{1/p} \operatorname{sgn} h(s) \; ds + \mathcal{C}, \; h \in C(\mathbb{R})$$

yields a solution $[f] = Ih$ of (6) and (1'). ■

3. Leibniz Rule

We now turn to the Leibniz rule operator equation

$$T(f \cdot g) = Tf \cdot g + f \cdot Tg, \quad f, g \in C^1(\mathbb{R}) \tag{7}$$

where $T : C^1(\mathbb{R}) \to C(\mathbb{R})$. It is known [KM1] that any operator T satisfying (7) has the form

$$Tf = b \; f' + a \; f \; \ln |f|, \quad f \in C^1(\mathbb{R}), \tag{8}$$

where $b, a \in C(\mathbb{R})$ and $0 \ln |0| := 0$. The results for the chain rule operator equation actually imply that the map T there is surjective. We will now study surjectivity conditions for T satisfying (7): Let $g \in C(\mathbb{R})$. We want to find $f \in C^1(\mathbb{R})$ with $Tf = g$. Then $Ig := f$ is a "generalized" integral in the Leibniz rule sense. We prove:

Proposition 3. *Assume $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ satisfies the Leibniz rule*

$$T(f_1 \cdot f_2) = Tf_1 \cdot f_2 + f_1 \cdot Tf_2; \quad f_1, f_2 \in C^1(\mathbb{R}) \; . \tag{7}$$

Suppose that for all $x \in \mathbb{R}$ there are $g_1, g_2 \in C^1(\mathbb{R})$ with $g_1(x) = g_2(x)$ and $(Tg_1)(x) \neq (Tg_2)(x)$. Then T is surjective, i.e. $Tf = g$ has a solution $f \in C^1(\mathbb{R})$ for any $g \in C(\mathbb{R})$.

P r o o f. Choose any $g \in C(\mathbb{R})$. To find $f \in C^1(\mathbb{R})$ with $Tf = g$, we have to solve the differential equation

$$Tf = b\ f' + a\ f\ \ \ln|f| = g$$

in $\mathbb{R}$, using (8). The assumption on T implies that $b(x) \neq 0$ for all $x \in \mathbb{R}$. Let $A := a/b$, $G := g/b$. Then $A, G \in C(\mathbb{R})$ and

$$f' + A\ f\ \ \ln|f| = G \tag{9}$$

has to be solved for a suitable $f \in C^1(\mathbb{R})$. Locally solutions of (9) exist; we only have to show that no singularity occurs on finite intervals. Assume that on some bounded open interval J we have that $f|_J \geq 2$ holds. Since A and G are continuous and bounded on $\overline{J}$, we conclude from (9) that there is a constant $M_j > 0$ such that $f' \leq M_J\ f\ \ \ln|f|$. The differential equation $F' = M_J\ F\ \ \ln|F|$, however, has a bounded solution in $\overline{J}$ since for $x_0 \in J$ and initial value $F(x_0) = f(x_0) \geq 2$

$$\begin{aligned} M_J(x - x_0) &= \int_{x_0}^{x} \frac{dF(t)}{F(t)\ln|F(t)|} = \int_{F(x_0)}^{F(x)} \frac{ds}{s\ln|s|} = \ln \frac{\ln\ F(x)}{\ln\ F(x_0)}, \\ F(x) &\leq F(x_0)^{\exp(M_J(x-x_0))}. \end{aligned}$$

By the generalized Gronwall inequality, cf. [H], Ch. III, Cor. 4.3, $|f| \leq |F|$ on $\overline{J}$. Therefore (9) admits a locally bounded solution $f \in C^1(\mathbb{R})$. A similar argument applies when $f|_J \leq -2$ holds. ■

We now claim that T is uniquely determined by its values Tf_1 and Tf_2 for two functions $f_1, f_2 \in C^1(\mathbb{R})$ for which there is no open interval in $\mathbb{R}$ such that either for some $c_1, c_2 \in \mathbb{R}$

$$\begin{aligned} \mathbb{R} \mid |f_1(x)|^{c_1} = |f_2(x)|^{c_2}\} \text{ or } \{x \in \mathbb{R} \mid f_1(x) \in \{0,1\}\} \\ \text{or } \{x \in \mathbb{R} \mid f_2(x) \in \{0,1\}\}. \end{aligned}$$

In this case

$$\begin{aligned} & \det \begin{pmatrix} f_1' & f_1 \ln|f_1| \\ f_2' & f_2 \ln|f_2| \end{pmatrix} = f_1' f_2 \ln|f_2| - f_2' f_1 \ln|f_1| \\ = &\ \ f_1 f_2[(\ln|f_1|)'(\ln|f_2|) - (\ln|f_2|)'(\ln|f_1|)] \\ = &\ \ (f_1 \ln|f_1|)(f_2 \ln|f_2|) \cdot [(\ln\ \ln|f_1|)' - (\ln\ \ln|f_2|)'] \ . \end{aligned}$$

If $(\ln\ \ln|f_1| - \ln\ \ln|f_2|)' = 0$ would hold on some open interval $I \subset \mathbb{R}$, we would get $\ln\ \ln|f_1| = \ln\ \ln|f_2| + \ln c$ for some constant $c > 0$ and hence $\ln|f_1| =$

$c \ln |f_2| = \ln |f_2|^c$, so that $|f_1| = |f_2|^c$ would be true. Hence the above determinant is non-zero in suitable points in arbitrarily small open intervals. If $g_1 = Tf_1$ and $g_2 = Tf_2$ are given, the continuous functions b and a in (10) are uniquely determined by the linear equations for $b(x)$ and $a(x)$,

$$b(x) f_j'(x) + a(x)\ f_j(x) \ln |f_j(x)| = g_j(x)$$

in points x where the above determinant is non-zero, and outside these points by a limiting argument using the continuity of b and a.

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