Journal of Mathematical Physics, Analysis, Geometry
2012, vol. 8, No. 1, pp. 21–37

Some Properties Concerning Curvature Tensors of Eight-Dimensional Walker Manifolds

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Received June 19, 2009

The main purpose of the present paper is to study conditions for the eight-dimensional Walker manifolds which admit a field of parallel null 4-planes to be Einsteinian, Osserman, or locally conformally flat.

Key words: Walker manifolds, Osserman manifolds, Riemannian extension, Einstein metric.

Mathematics Subject Classification 2000: 53C50 (primary); 53B05, 32Q20 (secondary).

1. Introduction

Let M_{2n} be a Riemannian manifold with a neutral metric, i.e., with a semi-Riemannian metric g of signature (n, n).We denote by $\Im_q^p(M_{2n})$ the set of all tensor fields of type (p, q) on M_{2n}. Manifolds, tensor fields and connections are always assumed to be differentiable of class C^∞.

By a Walker n-manifold, we mean a semi-Riemannian manifold which admits a field of parallel null r-planes with $r \leq \frac{n}{2}$. The canonical forms of the metrics were studied by Walker in [1]. Of special interest are even dimensional Walker manifolds $(n = 2m)$ admitting a field of null planes of maximum dimensionality $(r = m)$.

It is known that the Walker metrics have served as a powerful tool of constructing interesting indefinite metrics which exhibit various aspects of geometric properties not given by any positive definite metrics. Recently, it was shown [2, 3] that the Walker 4-manifolds of neutral signature admit a pair comprising an almost complex structure and an opposite almost complex structure, and that

Petean's nonflat indefinite Kähler-Einstein metric on a torus was obtained as an example of a Walker 4-manifold [4]. Moreover, an indefinite Ricci flat strictly almost Kähler metric on eight-dimensional torus was reported in [5]. Thus the Walker 4- and 8-manifolds display a large variety of indefinite geometry in four and eight-dimensions ([6–10]).

Our purpose is to systematically study the Walker metrics by focusing on their curvature properties. The main results of the present paper, the Walker metrics which are Einstein or locally conformally flat, are determined (Theorem 1, 2). To this end, the present paper is organized as follows. Section 2 develops some basic facts about Walker metrics. Their Levi–Civita connection and the curvature tensor are explicitly written for the purpose of the present analysis. In Section 3, the Walker metrics which are Einstein are studied, obtaining a family for the scalar curvature being zero (Theorem 1). In Section 4, the locally conformally flat Walker metrics are studied, and the examples of indefinite metrics of harmonic curvatures are exhibited. Finally, the Osserman property of Walker metrics in dimension eight are studied.

2. The canonical form of Walker metrics in dimension eight

A neutral g on an 8-manifold M_8 is said to be a Walker metric if there exists a 4-dimensional null distribution D on M_8 which is parallel with respect to g. From Walker theorem [1], there is a system of the coordinates $(x^1, \ldots, x^8)$ with respect to which g takes the local canonical form

$$g = (g_{ij}) = \begin{pmatrix} 0 & I_4 \\ I_4 & B \end{pmatrix}, \tag{2.1}$$

where I_4 is the unit 4×4 matrix and B is a 4×4 symmetric matrix whose entries are functions of the coordinates $(x^1, \ldots, x^8)$. Note that g is of neutral signature $(++++----)$, and that the parallel null 4-plane D is spanned locally by $\{\partial_1, \partial_2, \partial_3, \partial_4\}$, where $\partial_i = \frac{\partial}{\partial x^i}$, $(i = 1, \ldots, 8)$.

In this paper, we consider the specific Walker metrics on M_8 with B of the form

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

where $a,\ b$ are smooth functions of the coordinates $(x^1, \ldots, x^8)$. The Walker metrics with conditions (2.1) and (2.2) are studied by Matsushita et al. in [5]. There is a famous Goldberg conjecture, which states that the almost complex structure of a compact almost Kahler–Einstein Riemannian manifold is integrable. In [5], by considering the 8-dimensional Walker manifolds with conditions (2.1) and

(2.2), the authors have shown that the neutral-signature version of Goldberg's conjecture fails.

It is well known that if the Walker manifolds M_4 and M_8 are Einstein (or $*$-Einstein), then the Walker metrics on M_4 and M_8 can be viewed as Riemannian extensions from manifolds (M_2, ∇) and $(M_4, \widetilde{\nabla})$ to their cotangent bundles, respectively (see [11], [9] and Section 5 in the present paper). Moreover, let there be given on both manifolds M_2 and M_4 holomorphic (analytic) connections with respect to complex structures (for example, these structures naturally appeare in anti-Kähler geometry). Then (M_2, ∇) is locally flat [12, p. 113], but $(M_4, \widetilde{\nabla})$ is not locally flat. Hence the case M_8 with Walker metrics is of special interest and can be viewed as a Riemannian extension of non-flat holomorphic connection $\widetilde{\nabla}$. Indeed, if (M_2, ∇) is locally flat, then the Riemannian extension ${}^R\nabla$ to the cotangent bundle ${}^*T(M_2)$ has the components of the form

$$
{}^R\nabla = \begin{pmatrix} 0 & \delta^i_j \\ \delta^j_i & 0 \end{pmatrix},
$$

i.e. the metric ${}^R\nabla$ is trivial, and hence it is not localy isometric to the Walker manifold $(M_4, {}^w g)$ with conditions $a \neq 0$, $b \neq 0$, $c \neq 0$ (even if $a = b \neq 0$, $c = 0$), where

$$
{}^w g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}.
$$

A straightforward calculation using the fact that the inverse of the metric tensor, $g^{-1} = (g^{\alpha\beta})$, gives

$$
g^{-1} = \begin{pmatrix} -B & I_4 \\ I_4 & 0 \end{pmatrix}.
$$

The Levi–Civita connection of a Walker metric (2.1), with B given as in (2.2), is given by

$$
\begin{aligned}
\nabla_{\partial_1}\partial_5 &= \frac{1}{2}a_1\partial_1,\ \nabla_{\partial_1}\partial_7 = \frac{1}{2}b_1\partial_3,\ \nabla_{\partial_2}\partial_5 = \frac{1}{2}a_2\partial_1,\ \nabla_{\partial_2}\partial_7 = \frac{1}{2}b_2\partial_3, \\
\nabla_{\partial_3}\partial_5 &= \frac{1}{2}a_3\partial_1,\ \nabla_{\partial_3}\partial_7 = \frac{1}{2}b_3\partial_3,\ \nabla_{\partial_4}\partial_5 = \frac{1}{2}a_4\partial_1,\ \nabla_{\partial_4}\partial_7 = \frac{1}{2}b_4\partial_3, \\
\nabla_{\partial_5}\partial_5 &= \frac{1}{2}a_5\partial_1,\ \nabla_{\partial_5}\partial_6 = \frac{1}{2}a_6\partial_1,\ \nabla_{\partial_5}\partial_7 = \frac{1}{2}a_7\partial_1 + \frac{1}{2}b_5\partial_3, \\
\nabla_{\partial_5}\partial_8 &= \frac{1}{2}a_8\partial_1,\ \nabla_{\partial_6}\partial_7 = \frac{1}{2}b_6\partial_3,\ \nabla_{\partial_7}\partial_7 = \frac{1}{2}b_7\partial_3,\ \nabla_{\partial_7}\partial_8 = \frac{1}{2}b_8\partial_3, \quad (2.3)
\end{aligned}
$$

where a_i means a partial derivative $\frac{\partial}{\partial x^i}a(x^1, \dots, x^8)$.

Let R denote the Riemannian curvature tensor, taken with the sign convention $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. From (2.3), after a long but straightforward calculation we get that the nonzero components of the (0,4)-curvature tensor $R(X,Y,Z,V) = \langle R(X,Y)Z, V\rangle$ of any Walker metric (2.1) are determined by

$$\begin{aligned}
R_{1515} &= \frac{1}{2}a_{11},\ R_{1525} = \frac{1}{2}a_{12},\ R_{1535} = \frac{1}{2}a_{13},\ R_{1545} = \frac{1}{2}a_{14},\\
R_{1565} &= \frac{1}{2}a_{16},\ R_{1575} = \frac{1}{2}a_{17} - \frac{1}{4}a_3b_1,\ R_{1585} = \frac{1}{2}a_{18},\ R_{1717} = \frac{1}{2}b_{11},\\
R_{1727} &= \frac{1}{2}b_{12},\ R_{1737} = \frac{1}{2}b_{13},\ R_{1747} = \frac{1}{2}b_{14},\ R_{1757} = \frac{1}{2}b_{15} - \frac{1}{4}a_1b_1,\\
R_{1767} &= \frac{1}{2}b_{16},\ R_{1787} = \frac{1}{2}b_{18},\ R_{2525} = \frac{1}{2}a_{22},\ R_{2535} = \frac{1}{2}a_{23},\ R_{2545} = \frac{1}{2}a_{24},\\
R_{2565} &= \frac{1}{2}a_{26},\ R_{2575} = \frac{1}{2}a_{27} - \frac{1}{4}a_3b_2,\ R_{2585} = \frac{1}{2}a_{28},\ R_{2727} = \frac{1}{2}b_{22},\\
R_{2737} &= \frac{1}{2}b_{23},\ R_{2747} = \frac{1}{2}b_{24},\ R_{2757} = \frac{1}{2}b_{25} - \frac{1}{4}a_2b_1,\ R_{2767} = \frac{1}{2}b_{26},\\
R_{2787} &= \frac{1}{2}b_{28},\ R_{3535} = \frac{1}{2}a_{33},\ R_{3545} = \frac{1}{2}a_{34},\ R_{3565} = \frac{1}{2}a_{36},\\
R_{3575} &= \frac{1}{2}a_{37} - \frac{1}{4}a_3b_3,\ R_{3585} = \frac{1}{2}a_{38}, R_{3737} = \frac{1}{2}b_{33},\ R_{3747} = \frac{1}{2}b_{34},\\
R_{3757} &= \frac{1}{2}b_{35} - \frac{1}{4}a_3b_1,\ R_{3767} = \frac{1}{2}b_{36},\ R_{3787} = \frac{1}{2}b_{38},\ R_{4545} = \frac{1}{2}a_{44},\\
R_{4565} &= \frac{1}{2}a_{46},\ R_{4575} = \frac{1}{2}a_{47} - \frac{1}{4}a_3b_4,\ R_{4585} = \frac{1}{2}a_{48}, R_{4747} = \frac{1}{2}b_{44},\\
R_{4757} &= \frac{1}{2}b_{45} - \frac{1}{4}a_4b_1,\ R_{4767} = \frac{1}{2}b_{46},\ R_{4787} = \frac{1}{2}b_{48},\ R_{6565} = \frac{1}{2}a_{66},\\
R_{6575} &= \frac{1}{2}a_{67} - \frac{1}{4}a_3b_6,\ R_{6585} = \frac{1}{2}a_{68},\ R_{6767} = \frac{1}{2}b_{66},\ R_{6787} = \frac{1}{2}b_{68},\\
R_{7575} &= \frac{1}{2}a_{77} - \frac{1}{4}a_3b_7,\ R_{7585} = \frac{1}{2}a_{78} - \frac{1}{4}a_3b_8,\ R_{8585} = \frac{1}{2}a_{88},\ R_{8787} = \frac{1}{2}b_{88}.
\end{aligned} \tag{2.4}$$

3. Einstein-Walker metrics

We now turn our attention to the Einstein conditions for the Walker metric (2.1) with B given as in (2.2). As a matter of notation, let R_{ij} and Sc denote the Ricci tensor and the scalar curvature of a Walker metric (2.1). From equations (2.4), the nonzero components of the Ricci tensor R_{ij} are characterized by

$$\begin{aligned}
R_{15} &= \frac{1}{2}a_{11},\ R_{25} = \frac{1}{2}a_{12},\ R_{35} = \frac{1}{2}a_{13},\ R_{45} = \frac{1}{2}a_{14},\ R_{17} = \frac{1}{2}b_{13},\ R_{27} = \frac{1}{2}b_{23},\\
R_{37} &= \frac{1}{2}b_{33},\ R_{47} = \frac{1}{2}b_{34},\ R_{56} = \frac{1}{2}a_{16},\ R_{58} = \frac{1}{2}a_{18},\ R_{67} = \frac{1}{2}b_{36},\ R_{78} = \frac{1}{2}b_{38},\\
R_{57} &= \frac{1}{2}b_{35} - \frac{1}{2}a_3b_1 + \frac{1}{2}a_{17},\ R_{55} = -a_{26} - a_{37} - a_{48} + \frac{1}{2}(aa_{11} + ba_{33} + a_3b_3),\\
R_{77} &= -b_{15} - b_{26} - b_{48} + \frac{1}{2}(ab_{11} + a_1b_1 + bb_{33}).
\end{aligned} \tag{3.1}$$

From (2.1) and (3.1), the scalar curvature of a Walker metric (2.1) is given by

$$Sc = \sum_{i,j=1}^{8} g^{ij}R_{ij} = a_{11} + b_{33}. \tag{3.2}$$

R e m a r k 1. There exists a Walker metric with the prescribed scalar curvature, because from (3.2) we can always choose two suitable functions a and b.

Next we prove the main result in this section.

Theorem 1. *A Walker metric* (2.1) *is Einstein if and only if the defining functions $a(x^1,\dots,x^8)$ and $b(x^1,\dots,x^8)$ are as follows:*

$$\begin{cases} a(x^1,\dots,x^8) = a(x^1,x^3,x^5,x^7) = x^1R(x^5,x^7) + x^3S(x^5,x^7) + \xi(x^5,x^7)\\ b(x^1,\dots,x^8) = b(x^1,x^3,x^5,x^7) = x^3P(x^5,x^7) + x^1Q(x^5,x^7) + \eta(x^5,x^7), \end{cases} \tag{3.3}$$

where $\xi(x^5,x^7)$ and $\eta(x^5,x^7)$ are arbitrary smooth functions, while $P(x^5,x^7)$, $R(x^5,x^7)$, $S(x^5,x^7)$and $Q(x^5,x^7)$ are smooth functions satisfying

$$S_7 = \frac{1}{2}SP, Q_5 = \frac{1}{2}RQ, R_7 + P_5 = SQ. \tag{3.4}$$

P r o o f. The Einstein equations defined by $G_{ij} = R_{ij} - \frac{1}{8}Scg_{ij} = 0$ for a Walker metric (2.1) are as follows:

$$\text{(i)} \begin{cases} G_{25} = \frac{1}{2}a_{12} = 0, \quad G_{17} = \frac{1}{2}b_{13} = 0,\\ G_{35} = \frac{1}{2}a_{13} = 0, \quad G_{27} = \frac{1}{2}b_{23} = 0,\\ G_{45} = \frac{1}{2}a_{14} = 0, \quad G_{47} = \frac{1}{2}b_{34} = 0,\\ G_{56} = \frac{1}{2}a_{16} = 0, \quad G_{67} = \frac{1}{2}b_{36} = 0,\\ G_{58} = \frac{1}{2}a_{18} = 0, \quad G_{78} = \frac{1}{2}b_{38} = 0, \end{cases}$$

$$
\text{(ii)} \begin{cases} G_{26} = G_{48} = -\frac{1}{8}(a_{11} + b_{33}) = 0, \\ G_{15} = \frac{1}{8}(3a_{11} - b_{33}) = 0, \\ G_{37} = \frac{1}{8}(3b_{33} - a_{11}) = 0, \end{cases}
$$

$$
\text{(iii) } G_{57} = \tfrac{1}{2}(a_{17} - a_3 b_1 + b_{35}) = 0,
$$

$$
\text{(iv) } G_{55} = -a_{26} - a_{37} - a_{48} + \frac{1}{8}a(3a_{11} - b_{33}) + \frac{1}{2}(ba_{33} + a_3 b_3) = 0,
$$

$$
\text{(v) } G_{77} = -b_{15} - b_{26} - b_{48} + \frac{1}{2}(ab_{11} + a_1 b_1) + \frac{1}{8}b(3b_{33} - a_{11}) = 0. \tag{3.5}
$$

We divide the proof of this theorem into two steps.

Step 1. The (i) $PDEs$ in the Einstein conditions (3.5) imply that a and b take the following forms:

$$
\begin{cases} a(x^1, \ldots, x^8) = \bar{a}(x^1, x^5, x^7) + \hat{a}(x^2, x^3, x^4, x^5, x^6, x^7, x^8), \\ b(x^1, \ldots, x^8) = \bar{b}(x^3, x^5, x^7) + \hat{b}(x^1, x^2, x^4, x^5, x^6, x^7, x^8). \end{cases} \tag{3.6}
$$

From the (ii) $PDEs$ in the Einstein conditions (3.5), we get

$$
\begin{cases} a_{11} = 0, \\ b_{33} = 0. \end{cases} \tag{3.7}
$$

Substituting these functions a and b from (3.6) into (3.7), we get

$$
\begin{aligned} \bar{a}(x^1, x^5, x^7) &= x^1 R(x^5, x^7) + \xi(x^5, x^7) \\ \bar{b}(x^3, x^5, x^7) &= x^3 P(x^5, x^7) + \eta(x^5, x^7). \end{aligned}
$$

Therefore we have

$$
\begin{cases} a(x^1, \ldots, x^8) = x^1 R(x^5, x^7) + \hat{a}(x^2, x^3, x^4, x^5, x^6, x^7, x^8) \\ b(x^1, \ldots, x^8) = x^3 P(x^5, x^7) + \hat{b}(x^1, x^2, x^4, x^5, x^6, x^7, x^8) \end{cases} \tag{3.8}
$$

for some functions $R(x^5, x^7)$ and $P(x^5, x^7)$.

Step 2. The functions a and b in (3.8) satisfy the (i) and (ii) $PDEs$ in the Einstein conditions (3.5). We must consider further conditions for a and b to satisfy the (iii)–(v) $PDEs$ in (3.5). Inserting the functions a and b from (3.8) into the (iii) PDE in (3.5), we have that

$$
\hat{a}_3(x^2, x^3, x^4, x^5, x^6, x^7, x^8)\hat{b}_1(x^1, x^2, x^4, x^5, x^6, x^7, x^8) = R_7(x^5, x^7) + P_5(x^5, x^7).
$$

From this equation, we see that $\hat{a}_3$ does not depend on x^2, x^3, x^4, x^6 and x^8, and, similarly, $\hat{b}_1$ does not depend on x^1, x^2, x^4, x^6 and x^8. That is,

$\hat{a}(x^2, x^3, x^4, x^5, x^6, x^7, x^8) = x^3 S(x^5, x^7) + \xi(x^5, x^7)$, and $\hat{b}(x^1, x^2, x^4, x^5, x^6, x^7, x^8) = x^1 Q(x^5, x^7) + \eta(x^5, x^7)$. Hence, we can put

$$\begin{cases} a(x^1, \ldots, x^8) = x^1 R(x^5, x^7) + x^3 S(x^5, x^7) + \xi(x^5, x^7) \\ b(x^1, \ldots, x^8) = x^3 P(x^5, x^7) + x^1 Q(x^5, x^7) + \eta(x^5, x^7). \end{cases}$$

Finally, placing these expressions into the (iii)–(v) $PDEs$ in (3.5), we get

$$\begin{aligned} G_{57} &= \frac{1}{2}(R_7 - SQ + P_5) = 0, \\ G_{55} &= -S_7 + \frac{1}{2}S.P = 0, \\ G_{77} &= -Q_5 + \frac{1}{2}RQ = 0, \end{aligned}$$

which hold if and only if $R_7 + P_5 = SQ$, $S_7 = \frac{1}{2}SP$ and $Q_5 = \frac{1}{2}RQ$. This completes the proof. ■

Corollary 1. *Let Sc be the scalar curvature of the Einstein–Walker metric. Then the following equation holds:*

$$Sc = 0.$$

Now, we will analyze the Ricci flat property of the Einstein–Walker metric. For some special cases of the Einstein–Walker metric there can be written the following remarks.

R e m a r k 2. Let g be an Einstein–Walker metric. That is, the functions a and b of the metric g in (2.1) are given by the solution (3.5). g is necessarily of Ricci flat.

Case 1: $R = S = P = Q = 0$. Since there is no restriction on ξ and η, it follows that any Einstein–Walker metric of this case,

$$a = \xi(x^5, x^7), b = \eta(x^5, x^7),$$

is Ricci flat.

Case 2: $S = Q = 0$. In this case, the $PDEs$ in (3.4) reduce to $R_7 + P_5 = 0$, and thus the Einstein–Walker metric of the form

$$a = x^1 R(x^5, x^7) + \xi(x^5, x^7), b = x^3 P(x^5, x^7) + \eta(x^5, x^7)$$

is Ricci flat if and only if $R_7 + P_5 = 0$.

Case 3: $R = P = 0$. In this case, the $PDEs$ in (3.4) reduce to $S_7 = 0$, $Q_5 = 0$, $SQ = 0$. It immediately follows that $S(x^5, x^7) = S(x^5)$, $Q(x^5, x^7) =$

$Q(x^7)$. The condition $S(x^5,x^7)Q(x^5,x^7) = S(x^5)Q(x^7)$ holds if and only if either $S(x^5) = 0$ or $Q(x^7) = 0$. Consequently, the Ricci flat Einstein–Walker metric must be either

$$\begin{cases} a = \xi(x^5,x^7) \\ b = x^1Q(x^7) + \eta(x^5,x^7) \end{cases} \text{or} \begin{cases} a = x^3S(x^5) + \xi(x^5,x^7) \\ b = \eta(x^5,x^7). \end{cases}$$

R e m a r k 3. Let us assume that S and Q are nonzero. From the solutions of the first two $PDEs$ in (3.4), we can respectively write

$$S(x^5,x^7) = \gamma(x^5)e^{\frac{1}{2}\int P(x^5,x^7)dx^7}, Q = (x^5,x^7) = \delta(x^7)e^{\frac{1}{2}\int R(x^5,x^7)dx^5}$$

for some functions $\gamma(x^5)$, $\delta(x^7)$. If we choose $\alpha(x^5,x^7) = \int P(x^5,x^7)dx^7$ and $\beta(x^5,x^7) = \int R(x^5,x^7)dx^5$, then the a and b functions in (3.3) transform into

$$\begin{cases} a(x^1,\dots,x^8) = x^1\beta_5(x^5,x^7) + x^3\gamma(x^5)e^{\frac{1}{2}\alpha(x^5,x^7)} + \xi(x^5,x^7) \\ b(x^1,\dots,x^8) = x^3\alpha_7(x^5,x^7) + x^1\gamma(x^7)e^{\frac{1}{2}\beta(x^5,x^7)} + \eta(x^5,x^7). \end{cases}$$

Here, the third PDE in (3.4) reduces to

$$\beta_{57} + \alpha_{57} = \gamma\delta e^{\frac{1}{2}(\alpha+\beta)}$$

for any two functions α, β.

Let R denote the (0,4)-curvature tensor and consider its covariant derivative ∇R, whose vanishing is the condition for a metric to be locally symmetric. We complete this section by giving some results concerning the locally symmetric Ricci flat Einstein Walker metrics.

R e m a r k 4. Locally symmetric Ricci flat Walker metrics can be constructed from Remark 2. Such metrics are necessarily of Einstein as given in (3.3).

Assume $R = S = P = Q = 0$. Then the Ricci flat Einstein walker metric of the form

$$a = \xi(x^5,x^7), b = \eta(x^5,x^7)$$

is locally symmetric if and only if $\eta_{555} + \xi_{577} = 0$ and $\eta_{755} + \xi_{777} = 0$.

Assume $S = Q = 0$. Then it follows from equations (3.4) that $R_7 + P_5 = 0$. A locally symmetric Ricci flat Walker metric is of the form

$$a = x^1R(x^5) + \xi(x^5,x^7), b = x^3P(x^7) + \eta(x^5,x^7),$$

provided that the following two conditions hold:

$\eta_{555} + \xi_{577} = 0$ and $\eta_{755} + \xi_{777} = 0$.

Assume $R = P = 0$. There are two cases. In one case, we have that

$$a = \xi(x^5, x^7), b = x^1 Q(x^7) + \eta(x^5, x^7)$$

give a locally symmetric metric if and only if $2(\xi_{577} + \eta_{555}) = Q\xi_{55}$ and $2(\eta_{755} + \xi_{777}) = Q\xi_{75} - \xi_5 Q_7$. In the other case,

$$a = x^3 S(x^5) + \xi(x^5, x^7), b = \eta(x^5, x^7)$$

is locally symmetric if and only if $2(\eta_{555} + \xi_{577}) = S_5\eta_7 + S\eta_{57}$ and $2(\eta_{755} + \xi_{777}) = S\eta_{77}$.

4. Locally conformally flat Walker metrics

A semi-Riemannian manifold is locally conformally flat if and only if its Weyl tensor vanishes, where the Weyl tensor is given by

$$\begin{aligned} W(X,Y,Z,V) = R(X,Y,Z,V) &+ \tfrac{Sc}{(n-1)(n-2)}\left\{g(X,Z)g(Y,V) - g(X,V)g(Y,Z)\right\} \\ &- \tfrac{1}{(n-2)}\left\{Ric(X,Z)g(Y,V) - Ric(Y,Z)g(X,V)\right. \\ &\left.+ Ric(Y,V)g(X,Z) - Ric(X,V)g(Y,Z)\right\}. \end{aligned}$$

The nonzero components of Weyl tensor of a special Walker metric are given by

$$\begin{aligned}
W_{1525} &= \tfrac{7}{12}a_{12}, W_{1535} = \tfrac{7}{12}a_{13}, W_{1545} = \tfrac{7}{12}a_{14}, W_{1565} = \tfrac{7}{12}a_{16}, W_{1585} = \tfrac{7}{12}a_{18}, \\
W_{2525} &= \tfrac{1}{2}a_{22}, W_{2535} = \tfrac{1}{2}a_{23}, W_{2545} = \tfrac{1}{2}a_{24}, W_{2585} = \tfrac{1}{2}a_{28}, W_{3535} = \tfrac{1}{2}a_{33}, \\
W_{3545} &= \tfrac{1}{2}a_{34}, W_{3565} = \tfrac{1}{2}a_{36}, W_{3585} = \tfrac{1}{2}a_{38}, W_{4545} = \tfrac{1}{2}a_{44}, W_{4565} = \tfrac{1}{2}a_{46}, \\
W_{4585} &= \tfrac{1}{2}a_{48}, W_{6565} = \tfrac{1}{2}a_{66}, W_{6585} = \tfrac{1}{2}a_{68}, W_{8585} = \tfrac{1}{2}a_{88}, W_{1717} = \tfrac{1}{2}b_{11}, \\
W_{1727} &= \tfrac{1}{2}b_{12}, W_{1737} = \tfrac{7}{12}b_{13}, W_{1747} = \tfrac{1}{2}b_{14}, W_{1767} = \tfrac{1}{2}b_{16}, W_{1787} = \tfrac{1}{2}b_{18}, \\
W_{2727} &= \tfrac{1}{2}b_{22}, W_{2737} = \tfrac{1}{2}b_{23}, W_{2747} = \tfrac{1}{2}b_{24}, W_{2767} = \tfrac{1}{2}b_{26}, W_{2787} = \tfrac{1}{2}b_{28}, \\
W_{3747} &= \tfrac{7}{12}b_{34}, W_{3767} = \tfrac{7}{12}b_{36}, W_{3787} = \tfrac{7}{12}b_{38}, W_{4747} = \tfrac{1}{2}b_{44}, W_{4767} = \tfrac{1}{2}b_{46}, \\
W_{6767} &= \tfrac{1}{2}b_{66}, W_{6787} = \tfrac{1}{2}b_{68}, W_{8787} = \tfrac{1}{2}b_{88}, W_{1515} = \tfrac{1}{42}(27a_{11} - b_{33}), \\
W_{3737} &= \tfrac{1}{42}(27b_{33} - a_{11}), W_{4575} = \tfrac{1}{2}a_{47} - \tfrac{1}{4}a_3b_4 - \tfrac{1}{12}ab_{34}, \\
W_{2575} &= \tfrac{1}{2}a_{27} - \tfrac{1}{4}a_3b_2 - \tfrac{1}{12}ab_{23}, W_{2757} = \tfrac{1}{2}b_{25} - \tfrac{1}{4}a_2b_1 - \tfrac{1}{12}ba_{12}, \\
W_{4757} &= \tfrac{1}{2}b_{45} - \tfrac{1}{4}a_4b_1 - \tfrac{1}{12}ba_{14}, W_{5767} = \tfrac{1}{2}b_{56} - \tfrac{1}{4}a_6b_1 - \tfrac{1}{12}ba_{16}, \\
W_{5787} &= \tfrac{1}{2}b_{58} - \tfrac{1}{4}a_8b_1 - \tfrac{1}{12}ba_{18}, W_{6575} = \tfrac{1}{2}a_{67} - \tfrac{1}{4}a_3b_6 - \tfrac{1}{12}ab_{36}, \\
W_{7585} &= \tfrac{1}{2}a_{78} - \tfrac{1}{4}a_3b_8 - \tfrac{1}{12}ab_{38}, W_{1575} = \tfrac{7}{12}a_{17} - \tfrac{1}{3}a_3b_1 - \tfrac{1}{12}ab_{13} + \tfrac{1}{12}b_{35}, \\
W_{3757} &= \tfrac{7}{12}b_{35} - \tfrac{1}{3}a_3b_1 - \tfrac{1}{12}ba_{13} + \tfrac{1}{12}a_{17}, \\
W_{1757} &= \tfrac{2}{3}b_{15} + \tfrac{1}{6}b_{26} + \tfrac{1}{6}b_{48} - \tfrac{1}{12}ab_{11} - \tfrac{1}{3}a_1b_1 - \tfrac{5}{84}b(a_{11} - b_{33}),
\end{aligned}$$

$$\begin{aligned}
W_{2565} &= \tfrac{2}{3}a_{26} + \tfrac{1}{6}a_{37} + \tfrac{1}{6}a_{48} - \tfrac{1}{12}ba_{33} - \tfrac{1}{12}a_3b_3 - \tfrac{1}{84}a(5a_{11} - 2b_{33}),\\
W_{3575} &= \tfrac{2}{3}a_{37} + \tfrac{1}{6}a_{26} + \tfrac{1}{6}a_{48} - \tfrac{1}{12}ba_{33} - \tfrac{1}{3}a_3b_3 - \tfrac{5}{84}a(a_{11} + b_{33}),\\
W_{4787} &= \tfrac{2}{3}b_{48} + \tfrac{1}{6}b_{15} + \tfrac{1}{6}b_{26} - \tfrac{1}{12}ab_{11} - \tfrac{1}{12}a_1b_1 - \tfrac{1}{84}b(5b_{33} - 2a_{11}),\\
W_{5757} &= \tfrac{1}{2}b_{15} - \tfrac{1}{4}a_5b_1 + \tfrac{1}{6}b(a_{26} + a_{37} + a_{48}) + \tfrac{1}{6}a(b_{15} + b_{26} + b_{48})\\
&\quad -\tfrac{5}{84}ab(a_{11} + b_{33}) - \tfrac{1}{12}b(ba_{33} + a_3b_3) - \tfrac{1}{12}a(ab_{11} + a_1b_1),\\
W_{7575} &= \tfrac{1}{2}a_{77} - \tfrac{1}{4}a_3b_7 + \tfrac{1}{6}b(a_{26} + a_{37} + a_{48}) + \tfrac{1}{6}a(b_{15} + b_{26} + b_{48})\\
&\quad -\tfrac{5}{84}ab(a_{11} + b_{33}) - \tfrac{1}{12}b(ba_{33} + a_3b_3) - \tfrac{1}{12}a(ab_{11} + a_1b_1). \qquad (4.1)
\end{aligned}$$

Now it is possible to obtain the form of a locally conformally flat Walker metric as follows.

Theorem 2. *A Walker metric* (2.1) *is locally conformally flat if and only if the defining functions* $a = a(x^1, \ldots, x^8)$ *and* $b = b(x^1, \ldots, x^8)$ *satisfy*

$$\begin{cases} a = x^1K(x^5, x^7) + x^2T(x^5, x^7) + x^3L(x^5) + x^4R(x^5, x^7)\\ \quad + x^6F(x^5, x^7) + x^8D(x^5, x^7) + \xi(x^5, x^7)\\ b = x^1S(x^7) + x^2Q(x^5, x^7) + x^3P(x^5, x^7) + x^4M(x^5, x^7)\\ \quad + x^6N(x^5, x^7) + x^8V(x^5, x^7) + \eta(x^5, x^7) \end{cases} \qquad (4.2)$$

for any functions $K(x^5, x^7)$, $T(x^5, x^7)$, $L(x^5)$, $R(x^5, x^7)$, $F(x^5, x^7)$, $D(x^5, x^7)$, $\xi(x^5, x^7)$, $S(x^7)$, $Q(x^5, x^7)$, $P(x^5, x^7)$, $M(x^5, x^7)$, $N(x^5, x^7)$, $V(x^5, x^7)$, $\eta(x^5, x^7)$, *satisfying*

$$\begin{aligned}
K_7 &= \tfrac{1}{2}LS, T_7 = \tfrac{1}{2}LQ, R_7 = \tfrac{1}{2}LM, F_7 = \tfrac{1}{2}LN, D_7 = \tfrac{1}{2}LV, \xi_7 = \tfrac{1}{2}L\eta,\\
Q_5 &= \tfrac{1}{2}ST, P_5 = \tfrac{1}{2}SL, M_5 = \tfrac{1}{2}SR, N_5 = \tfrac{1}{2}SF, V_5 = \tfrac{1}{2}SD, \eta_5 = \tfrac{1}{2}S\xi,\\
KS &= 0, LP = 0. \qquad (4.3)
\end{aligned}$$

P r o o f. Since an eight-dimensional manifold is locally conformally flat if and only if the Weyl tensor vanishes, we consider (4.1) as a system of $PDEs$. We will prove this theorem in three steps.

Step 1. Considering the following components of the Weyl tensor of (4.1), we have

$$\begin{cases} W_{1525} = \frac{7}{12}a_{12} = 0\\ W_{1535} = \frac{7}{12}a_{13} = 0\\ W_{1545} = \frac{7}{12}a_{14} = 0\\ W_{1565} = \frac{7}{12}a_{16} = 0\\ W_{1585} = \frac{7}{12}a_{18} = 0 \end{cases} \quad \begin{cases} W_{1727} = \frac{1}{2}b_{12} = 0\\ W_{1737} = \frac{7}{12}b_{13} = 0\\ W_{1747} = \frac{1}{2}b_{14} = 0\\ W_{1767} = \frac{1}{2}b_{16} = 0\\ W_{1787} = \frac{1}{2}b_{18} = 0. \end{cases} \qquad (4.4)$$

First, the $PDEs$ (4.4) imply that a and b take the form

$$\begin{cases} a = a(x^1, \ldots, x^8) = \bar{a}(x^1, x^5, x^7) + \hat{a}(x^2, x^3, x^4, x^5, x^6, x^7, x^8)\\ b = b(x^1, \ldots, x^8) = \bar{b}(x^1, x^5, x^7) + \hat{b}(x^2, x^3, x^4, x^5, x^6, x^7, x^8). \end{cases} \qquad (4.5)$$

From the two $PDEs$, $W_{1515} = \frac{1}{42}(27a_{11} - b_{33}) = 0$ and $W_{3737} = \frac{1}{42}(27b_{33} - a_{11}) = 0$, we obtain

$$\begin{cases} a_{11} = 0 \\ b_{33} = 0. \end{cases}$$

Then we have $a_{11} = 0$ so $\bar{a}_{11} = 0$, then $\bar{a}(x^1, x^5, x^7) = x^1 K(x^5, x^7) + \xi(x^5, x^7)$, moreover, $b_{11} = 0$ so $\bar{b}_{11} = 0$, then $\bar{b}(x^1, x^5, x^7) = x^1 S(x^5, x^7) + \eta(x^5, x^7)$. Therefore equations (4.5) transform into

$$\begin{cases} a = x^1 K(x^5, x^7) + \hat{a}(x^2, x^3, x^4, x^5, x^6, x^7, x^8) \\ b = x^1 S(x^5, x^7) + \hat{b}(x^2, x^3, x^4, x^5, x^6, x^7, x^8). \end{cases} \tag{4.6}$$

Substituting the functions a and b from (4.6) into the following $PDEs$:

$$\begin{cases} W_{2535} = \frac{1}{2}a_{23} = 0 \\ W_{2545} = \frac{1}{2}a_{24} = 0 \\ W_{2585} = \frac{1}{2}a_{28} = 0, \end{cases} \quad \begin{cases} W_{2737} = \frac{1}{2}b_{23} = 0 \\ W_{2747} = \frac{1}{2}b_{24} = 0 \\ W_{2767} = \frac{1}{2}b_{26} = 0 \\ W_{2787} = \frac{1}{2}b_{28} = 0, \end{cases}$$

the functions a and b become of the forms

$$\begin{cases} \hat{a}(x^2, x^3, x^4, x^5, x^6, x^7, x^8) = \bar{\hat{a}}(x^2, x^5, x^6, x^7) + \hat{\hat{a}}(x^3, x^4, x^5, x^6, x^7, x^8) \\ \hat{b}(x^2, x^3, x^4, x^5, x^6, x^7, x^8) = \bar{\hat{b}}(x^2, x^5, x^7) + \hat{\hat{b}}(x^3, x^4, x^5, x^6, x^7, x^8). \end{cases}$$

Since $W_{2525} = \frac{1}{2}a_{22} = 0$ and $W_{2727} = \frac{1}{2}b_{22} = 0$, we have that $\bar{\hat{a}}(x^2, x^5, x^6, x^7) = x^2 T(x^5, x^6, x^7) + \xi(x^5, x^6, x^7)$ and $\bar{\hat{b}}(x^2, x^5, x^7) = x^2 Q(x^5, x^7) + \eta(x^5, x^7)$. Hence we can put

$$\begin{cases} a = x^1 K(x^5, x^7) + x^2 T(x^5, x^6, x^7) + \hat{\hat{a}}(x^3, x^4, x^5, x^6, x^7, x^8) \\ b = x^1 S(x^5, x^7) + x^2 Q(x^5, x^7) + \hat{\hat{b}}(x^3, x^4, x^5, x^6, x^7, x^8). \end{cases} \tag{4.7}$$

On inserting the functions a and b from (4.7) into the following $PDEs$:

$$\begin{cases} W_{3545} = \frac{1}{2}a_{34} = 0 \\ W_{3565} = \frac{1}{2}a_{36} = 0 \\ W_{3585} = \frac{1}{2}a_{38} = 0, \end{cases} \quad \begin{cases} W_{3747} = \frac{7}{12}b_{34} = 0 \\ W_{3767} = \frac{7}{12}b_{36} = 0 \\ W_{3787} = \frac{7}{12}b_{38} = 0, \end{cases}$$

we get

$$\begin{cases} \hat{\hat{a}}(x^3, x^4, x^5, x^6, x^7, x^8) = \bar{\hat{\hat{a}}}(x^3, x^5, x^7) + \hat{\hat{\hat{a}}}(x^4, x^5, x^6, x^7, x^8) \\ \hat{\hat{b}}(x^3, x^4, x^5, x^6, x^7, x^8) = \bar{\hat{\hat{b}}}(x^3, x^5, x^7) + \hat{\hat{\hat{b}}}(x^4, x^5, x^6, x^7, x^8). \end{cases}$$

From $W_{3535} = \frac{1}{2}a_{33} = 0$ and $b_{33} = 0$, it can be written that $\bar{\hat{\hat{a}}}(x^3, x^5, x^7) = x^3L(x^5, x^7) + \xi(x^5, x^7)$ and $\bar{\hat{\hat{b}}}(x^3, x^5, x^7) = x^3P(x^5, x^7) + \eta(x^5, x^7)$. Therefore, the functions a and b take the form

$$\begin{cases} a = x^1K(x^5, x^7) + x^2T(x^5, x^6, x^7) + x^3L(x^5, x^7) + \hat{\hat{a}}(x^4, x^5, x^6, x^7, x^8) \\ b = x^1S(x^5, x^7) + x^2Q(x^5, x^7) + x^3P(x^5, x^7) + \hat{\hat{b}}(x^4, x^5, x^6, x^7, x^8). \end{cases} \tag{4.8}$$

Analogously, the functions a and b, satisfying the following $PDEs:$

$$\begin{cases} W_{4545} = \frac{1}{2}a_{44} = 0 \\ W_{4565} = \frac{1}{2}a_{46} = 0 \\ W_{4585} = \frac{1}{2}a_{48} = 0, \end{cases} \quad \begin{cases} W_{4747} = \frac{1}{2}b_{44} = 0 \\ W_{4767} = \frac{1}{2}b_{46} = 0, \end{cases}$$

$$\begin{cases} W_{6565} = \frac{1}{2}a_{66} = 0 \\ W_{6585} = \frac{1}{2}a_{68} = 0 \\ W_{8585} = \frac{1}{2}a_{88} = 0, \end{cases} \quad \begin{cases} W_{6767} = \frac{1}{2}b_{66} = 0 \\ W_{6787} = \frac{1}{2}b_{68} = 0 \\ W_{8787} = \frac{1}{2}b_{88} = 0, \end{cases}$$

take the form

$$\begin{cases} a = x^1K(x^5, x^7) + x^2T(x^5, x^6, x^7) + x^3L(x^5, x^7) + x^4R(x^5, x^7) \\ \quad + x^6F(x^5, x^7) + x^8D(x^5, x^7) + \xi(x^5, x^7) \\ b = x^1S(x^5, x^7) + x^2Q(x^5, x^7) + x^3P(x^5, x^7) + x^4M(x^5, x^7, x^8) \\ \quad + x^6N(x^5, x^7) + x^8V(x^5, x^7) + \eta(x^5, x^7) \end{cases}$$

for any functions $K(x^5, x^7)$, $T(x^5, x^7)$, $L(x^5)$, $R(x^5, x^7)$, $F(x^5, x^7)$, $D(x^5, x^7)$, $\xi(x^5, x^7)$, $S(x^7)$, $Q(x^5, x^7)$, $P(x^5, x^7)$, $M(x^5, x^7)$, $N(x^5, x^7)$, $V(x^5, x^7)$, $\eta(x^5, x^7)$.

Step 2. Considering the result of step one, placing this result into the following $PDEs$:

$$W_{2575} = \frac{1}{2}(a_{27} - \frac{1}{2}a_3b_2 - \frac{1}{6}ab_{23}) = 0, W_{4757} = \frac{1}{2}(b_{45} - \frac{1}{2}a_4b_1 - \frac{1}{6}ba_{14}) = 0,$$

there can be obtained the following conditions:

$$T_7 = \frac{1}{2}LQ, M_5 = \frac{1}{2}SR. \tag{4.9}$$

From (4.9), we can see that T does not depend on x^6, and M does not depend on x^8. Therefore we get

$$\begin{cases} a = x^1K(x^5, x^7) + x^2T(x^5, x^6, x^7) + x^3L(x^5, x^7) + x^4R(x^5, x^7) \\ \quad + x^6F(x^5, x^7) + x^8D(x^5, x^7) + \xi(x^5, x^7). \\ b = x^1S(x^5, x^7) + x^2Q(x^5, x^7) + x^3P(x^5, x^7) + x^4M(x^5, x^7, x^8) \\ \quad + x^6N(x^5, x^7) + x^8V(x^5, x^7) + \eta(x^5, x^7). \end{cases} \tag{4.10}$$

Inserting the functions a and b from (4.10) into to the following $PDEs$:

$$\begin{aligned}
W_{4575} &= \tfrac{1}{2}(a_{47} - \tfrac{1}{2}a_3b_4 - \tfrac{1}{6}ab_{34}) = 0, W_{2757} = \tfrac{1}{2}(b_{25} - \tfrac{1}{2}a_2b_1 - \tfrac{1}{6}ba_{12}) = 0,\\
W_{5767} &= \tfrac{1}{2}(b_{56} - \tfrac{1}{2}a_6b_1 - \tfrac{1}{6}ba_{16}) = 0, W_{5787} = \tfrac{1}{2}(b_{58} - \tfrac{1}{2}a_8b_1 - \tfrac{1}{6}ba_{18}) = 0,\\
W_{6575} &= \tfrac{1}{2}(a_{67} - \tfrac{1}{2}a_3b_6 - \tfrac{1}{6}ab_{36}) = 0, W_{7585} = \tfrac{1}{2}(a_{78} - \tfrac{1}{2}a_3b_8 - \tfrac{1}{6}ab_{38}) = 0,\\
W_{1575} &= \tfrac{1}{12}(7a_{17} - 4a_3b_1 - ab_{13} + b_{35}) = 0,\\
W_{3757} &= \tfrac{1}{12}(7b_{35} - 4a_3b_1 - ba_{13} + a_{17}) = 0,\\
W_{1757} &= \tfrac{2}{3}b_{15} + \tfrac{1}{6}b_{26} + \tfrac{1}{6}b_{48} - \tfrac{1}{12}ab_{11} - \tfrac{1}{3}a_1b_1 - \tfrac{5}{84}b(a_{11} - b_{33}) = 0,\\
W_{4787} &= \tfrac{2}{3}b_{48} + \tfrac{1}{6}b_{15} + \tfrac{1}{6}b_{26} - \tfrac{1}{12}ab_{11} - \tfrac{1}{12}a_1b_1 - \tfrac{1}{84}b(5b_{33} - 2a_{11}) = 0,\\
W_{2565} &= \tfrac{2}{3}a_{26} + \tfrac{1}{6}a_{37} + \tfrac{1}{6}a_{48} - \tfrac{1}{12}ba_{33} - \tfrac{1}{12}a_3b_3 - \tfrac{1}{84}a(5a_{11} - 2b_{33}) = 0,\\
W_{3575} &= \tfrac{2}{3}a_{37} + \tfrac{1}{6}a_{26} + \tfrac{1}{6}a_{48} - \tfrac{1}{12}ba_{33} - \tfrac{1}{3}a_3b_3 - \tfrac{5}{84}a(a_{11} + b_{33}) = 0,
\end{aligned}$$

we obtain, respectively, the following conditions:

$$\begin{aligned}
R_7 &= \tfrac{1}{2}LM, Q_5 = \tfrac{1}{2}ST, N_5 = \tfrac{1}{2}SF, V_5 = \tfrac{1}{2}SD,\\
F_7 &= \tfrac{1}{2}LN, D_7 = \tfrac{1}{2}LV, K_7 = P_5 = \tfrac{1}{2}LS, S_5 = \tfrac{1}{2}KS,\\
L_7 &= \tfrac{1}{2}LP.
\end{aligned}$$

Step 3. It follows from the previous steps that the last two equations (4.1) reduce to the equations

$$\begin{aligned}
W_{5757} = \tfrac{1}{2}b_{15} - \tfrac{1}{4}a_5b_1 + \tfrac{1}{6}b(a_{26} + a_{37} + a_{48}) + \tfrac{1}{6}a(b_{15} + b_{26} + b_{48})\\
- \tfrac{5}{84}ab(a_{11} + b_{33}) - \tfrac{1}{12}b(ba_{33} + a_3b_3) - \tfrac{1}{12}a(ab_{11} + a_1b_1) = 0,
\end{aligned}$$

$$\begin{aligned}
W_{7575} = \tfrac{1}{2}a_{77} - \tfrac{1}{4}a_3b_7 + \tfrac{1}{6}b(a_{26} + a_{37} + a_{48}) + \tfrac{1}{6}a(b_{15} + b_{26} + b_{48})\\
- \tfrac{5}{84}ab(a_{11} + b_{33}) - \tfrac{1}{12}b(ba_{33} + a_3b_3) - \tfrac{1}{12}a(ab_{11} + a_1b_1) = 0.
\end{aligned} \tag{4.11}$$

From the first equation in (4.11), we conclude that

$$S = S(x^7), \eta_5 = \tfrac{1}{2}S\xi. \tag{4.12}$$

From the first equation in (4.12) and $S_5 = \frac{1}{2}KS$, we find

$$S_5 = 0, \tag{4.13}$$

that is, $KS = 0$. Similarly, from the second equation in (4.11) we conclude that

$$L = L(x^5), \xi_7 = \tfrac{1}{2}L\eta. \tag{4.14}$$

From the first equation in (4.14) and $L_7 = \frac{1}{2}LP$, we obtain

$$L_7 = 0, \tag{4.15}$$

that is, $LP = 0$. Therefore, from (4.10), (4.13) and (4.15), we have that

$$\begin{cases} a = x^1K(x^5,x^7) + x^2T(x^5,x^7) + x^3L(x^5) + x^4R(x^5,x^7) \\ \quad + x^6F(x^5,x^7) + x^8D(x^5,x^7) + \xi(x^5,x^7) \\ b = x^1S(x^7) + x^2Q(x^5,x^7) + x^3P(x^5,x^7) + x^4M(x^5,x^7) \\ \quad + x^6N(x^5,x^7) + x^8V(x^5,x^7) + \eta(x^5,x^7), \end{cases}$$

which finishes the proof. ■

Let R_{ij} and Sc denote the Ricci tensor and the scalar curvature of a Walker metric (2.1). It follows from the equation expression for the components of the curvature tensor that the Ricci operator $\langle \hat{R}ic(X), Y\rangle = Ric(X,Y)$ satisfies

$$\begin{aligned} \hat{R}ic(\partial_1) &= \tfrac{1}{2}a_{11}\partial_1 + \tfrac{1}{2}b_{13}\partial_3, \quad \hat{R}ic(\partial_2) = \tfrac{1}{2}a_{12}\partial_1 + \tfrac{1}{2}b_{23}\partial_3, \\ \hat{R}ic(\partial_3) &= \tfrac{1}{2}a_{13}\partial_1 + \tfrac{1}{2}b_{33}\partial_3, \quad \hat{R}ic(\partial_4) = \tfrac{1}{2}a_{14}\partial_1 + \tfrac{1}{2}b_{34}\partial_3, \\ \hat{R}ic(\partial_6) &= \tfrac{1}{2}a_{16}\partial_1 + \tfrac{1}{2}b_{36}\partial_3, \quad \hat{R}ic(\partial_8) = \tfrac{1}{2}a_{18}\partial_1 + \tfrac{1}{2}b_{38}\partial_3, \\ \hat{R}ic(\partial_5) &= \tfrac{1}{2}(-2a_{26} - 2a_{37} - 2a_{48} + ba_{33} + a_3b_3)\partial_1 + \tfrac{1}{2}a_{16}\partial_2 \\ &\quad + \tfrac{1}{2}(b_{35} - a_3b_1 + a_{17} - ba_{13})\partial_3 + \tfrac{1}{2}a_{18}\partial_4 + \tfrac{1}{2}a_{11}\partial_5 \\ &\quad + \tfrac{1}{2}a_{12}\partial_6 + \tfrac{1}{2}a_{13}\partial_7 + \tfrac{1}{2}a_{14}\partial_8, \\ \hat{R}ic(\partial_7) &= \tfrac{1}{2}(b_{35} - a_3b_1 + a_{17} - ab_{13})\partial_1 + \tfrac{1}{2}b_{36}\partial_2 + \\ &\quad + \tfrac{1}{2}(-2b_{15} - 2b_{26} - 2b_{48} + ab_{11} + a_1b_1)\partial_3 \\ &\quad + \tfrac{1}{2}b_{38}\partial_4 + \tfrac{1}{2}b_{13}\partial_5 + \tfrac{1}{2}b_{23}\partial_6 + \tfrac{1}{2}b_{33}\partial_7 + \tfrac{1}{2}b_{34}\partial_8. \end{aligned} \tag{4.16}$$

The result follows by a straightforward calculation.

R e m a r k 5. Considering (4.16), we see that the locally conformally flat Walker metric satisfying (4.2) has a vanishing Ricci operator.

R e m a r k 6. Considering (3.2) and (4.2), we see that the locally conformally flat Walker metric (4.2) has a vanishing scalar curvature.

From Remark 6 we have

R e m a r k 7. Locally conformally flat Walker metrics are semi-Riemannian manifolds with harmonic curvature [13].

5. Osserman–Walker 8-manifolds

Let M_n be an n-dimensional differentiable manifold of class C^∞, ${}^CT(M_n)$ be its cotangent bundle, and π, the natural projection ${}^CT(M_n) \to M_n$. A system of local coordinates $(U; x^i)$, $i = 1, \ldots, n$ in M_n induces on ${}^CT(M_n)$ a system of local coordinates $(\pi^{-1}(U); x^i, x^{\bar{\imath}} = p_i)$, $i = 1, \ldots, n$, $\bar{\imath} = n + i = n + 1, \ldots, 2n$, where $x^{\bar{i}} = p_i$ is the cartesian coordinates of covectors p in each cotangent spaces ${}^C T_x(M_n)$, $x \in U$ with respect to the natural coframe $\{dx^i\}$. We denote by $\Im^r_s(M_n)$ $(\Im^r_s({}^C T(M_n)))$ the module over $F(M_n)$ $(F({}^C T(M_n)))$ of C^∞ tensor field of type (r,s), where $F(M_n)$ $(F({}^C T(M_n)))$ is the ring of the real-valued C^∞functions of M_n $({}^C T(M_n))$.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be the local expression in $U \subset M_n$ of a vector field $X \in \Im^1_0(M_n)$, and a 1-form $\omega \in \Im^0_1(M_n)$, respectively. Then the horizontal lift ${}^H X \in \Im^1_0({}^C T(M_n))$ of X and the vertical lift ${}^V \omega \in \Im^1_0({}^C T(M_n))$ of ω are given, respectively, by

$$ {}^H X = X^i \frac{\partial}{\partial x^i} + \sum_i p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial x^{\bar{i}}} \tag{5.1} $$

and

$$ {}^V \omega = \sum_i \omega_i \frac{\partial}{\partial x^{\bar{i}}} \tag{5.2} $$

with respect to the natural frame $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\right\}$, where Γ^h_{ij} are the components of a symmetric (torsion-free) affine connection ∇ on M_n.
We now consider a tensor field ${}^R\nabla \in \Im^0_2({}^C T(M_n))$, whose components in $\pi^{-1}(U)$ are given by

$$ {}^R\nabla = ({}^R\nabla_{JI}) = \begin{pmatrix} -2p_h\Gamma^h_{ji} & \delta^i_j \\ \delta^j_i & 0 \end{pmatrix} \tag{5.3} $$

with respect to the natural frame, where δ^i_j denotes the Kronecker delta. The indices $I, J, K, \ldots = 1, \ldots, 2n$ indicate the indices with respect to the natural frame $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\right\}$. This tensor field defines a pseudo-Rimannian metric in ${}^C T(M_n)$, and the line element of the pseudo-Riemannian metric ${}^R\nabla$ is given by

$$ ds^2 = 2dx^i \delta p_i, $$

where $\delta p_i = dp_i - p_h \Gamma^h_{ji} dx^i$. This metric is called the Riemannian extension of the symmetric affine connection ∇ [14, p. 268].

The complete lift of a vector field $X \in \Im^1_0(M_n)$ to the cotangent bundle ${}^C T(M_n)$ is defined by

$$ {}^C X = X^i \frac{\partial}{\partial x^i} - \sum_i p_h \partial_i X^h \frac{\partial}{\partial x^{\bar{i}}}. \tag{5.4} $$

Using (5.3) and (5.4), we easily see that

$$ {}^R\nabla({}^C X, {}^C Y) = -\gamma(\nabla_X Y + \nabla_Y X), \tag{5.5} $$

where

$$ \gamma(\nabla_X Y + \nabla_Y X) = p_h(X^i \nabla_i Y^h + Y^i \nabla_i X^h). $$

Since the tensor field ${}^{R}\nabla \in \Im_2^0({}^{C}T(M_n))$ is completely determined by action on vector fields of type ${}^{C}X$ and ${}^{C}Y$ (see Proposition 4.2 of [14, p. 237]), we have an alternative definition of ${}^{R}\nabla$: the tensor field ${}^{R}\nabla$ is completely determined by condition (5.5).

From Theorem 1, we see that the metrics (3.3) can be viewed as the Riemannian extensions, i.e. they are locally isometric to the cotangent bundle ${}^{C}T(M_n)$ with metric ${}^{R}\nabla + \pi^{*}G$, where $\pi^{*}G$ is the pull-back on ${}^{C}T(M_n)$ of a symmetric tensor field $G \in \Im_2^0(M_n)$. On the other hand, we can be able to state the $PDEs$ in Theorem 1 in terms of the geometry of the corresponding torsion-free connection on the 4-dimensional base manifold. The condition (3.4) in Theorem 1 is just the condition to be affine Osserman on the base 4-manifold [11]. In [15], it is proved that $({}^{C}T(M_n), {}^{R}\nabla)$ is a pseudo-Riemannian Osserman space if and only if (M_n, ∇) is an affine Osserman space.

Thus we have the following theorem.

Theorem 3. *A Walker metric* (2.1) *is Osserman if and only if it is Einstein with the metric given by* (3.3) *and* (3.4).

6. Conclusions

A Walker n-manifold is a semi-Riemannian manifold which admits a field of parallel null r planes with $r \leq \frac{n}{2}$. In this article, we study the curvature properties of a Walker 8-manifold (M, g) which admits a field of parallel null 4-palnes. The metric g is necessarily of neutral signature $(++++----)$. In [5], the authors consider Goldberg's conjecture but for the metrics with neutral signature. They initially display examples of almost Kahler–Einstein neutral structures on R^8 such that the almost complex structure is not integrable. Then, they obtain the structures of the same type on the torus T^8. Therefore, it is proved that the neutral version of Goldberg's conjecture fails. For such restricted Walker 8-manifolds, we study mainly the curvature properties, e.g., the conditions for a Walker metric to be Einstein, Osserman, or locally conformally flat, etc. One of our main results is the exact solutions to the Einstein equations for a restricted Walker 8-manifold. The Walker metrics (3.3) can be alternatively described in terms of Riemannian extensions. As a particular case, when we consider that the base manifold is a four-dimensional Walker manifold, the condition (3.4) in Theorem 1 is just the condition to be affine Osserman on the base four-dimensional Walker manifolds [11]. Our another main result relates to the Osserman property of the Walker metrics (3.3).

Acknowledgement The authors are grateful to Professor E. Garcia Rio for valuable comments. The first and third authors acknowledge the support of the project 108T590 (TUBITAK, TURKEY).

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