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Narrow Operators on Bochner L_1-Spaces

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Narrow operators on $L_1(\mu, X)$-spaces are studied. We present a sufficient criterion for such an operator to be narrow that resembles the characterization on narrow operators on $L_1(\mu)$ and show that the criterion is necessary for certain X, e.g., for spaces with the RNP.

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1. Introduction

In this paper the letters X, Y, E will be used for real Banach spaces, μ and ν will be used for finite σ-additive measures on a σ-algebra Σ of subsets of a fixed set Ω. By $L_1(\mu, X)$ we denote the space of X-valued Bochner integrable functions on Ω; by $L_1(A, \mu, X)$ we denote the subspace of $L_1(\mu, X)$, consisting of functions supported on A. We denote the closed unit ball of a Banach space X by $B(X)$ and its unit sphere by $S(X)$.

A Banach space X is said to have the *Daugavet property* [7] if every rank-1 operator $T\colon X \to X$ satisfies

$$\|\mathrm{Id} + T\| = 1 + \|T\|. \tag{1.1}$$

In [8], an approach to the study of the class of operators satisfying (1.1) using the notion of a narrow operator was suggested, built on precursors from [5] and [10].

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Let us give the definition of a narrow operator.

Definition 1.1. *The (open) slice of $S(X)$ determined by a functional $x^* \in S(X^*)$ and $\varepsilon > 0$ is the set*

$$S(x^*, \varepsilon) = \{x \in S(X) \colon x^*(x) > 1 - \varepsilon\}.$$

Note that $S(x^*, \varepsilon) \subset S(X)$.

Definition 1.2. *An operator $T \colon X \to E$ is said to be narrow if for every $x, y \in S(X)$, every $\varepsilon > 0$ and every slice $S(x^*, \varepsilon)$ containing x there is a $z \in S(x^*, \varepsilon)$ such that $\|T(z - x)\| < \varepsilon$ and $\|z + y\| > 2 - \varepsilon$.*

Remark 1.3. *It was proved in [8] that instead of the slice $S(x^*, \varepsilon)$ in the definition above one can also take the intersection of an arbitrary weak neighbourhood of x with the sphere in the definition above.*

For $X = L_1(\mu)$ the following characterization of narrow operators was proved in [8, Th. 6.1].

Definition 1.4. *Let (Ω, Σ, μ) be an atomless probability space. A function $f \in L_1 = L_1(\mu)$ is said to be a balanced ε-peak on $A \in \Sigma$ if there is a subset $A_1 \subset A$ with $\mu(A_1) < \varepsilon$ such that*

1. $f = -1$ *for* $t \in A \setminus A_1$, $\operatorname{supp} f \subset A$,
2. $f \geq -1$,
3. $\int_\Omega f \, d\mu = 0$.

Theorem 1.5. *An operator $T \colon L_1(\mu) \to E$ is narrow if and only if for every $\varepsilon > 0$, and for every $A \in \Sigma$ there exists such a balanced ε-peak f on A that $\|T(f)\| < \varepsilon$.*

One can find more about the characterization of narrow operators on $L_1(\mu)$ as well as open problems in [6].

In the present paper we study narrow operators on the space $L_1(\mu, X)$ of the vector-valued Bochner integrable functions. It is proved that for a wide class of spaces X the narrow operators allow a description similar to Th. 1.5. At the same time there are spaces where the analogous description of narrow operators does not hold. Similar investigations about operators on $C(K, X)$-spaces were made in [2].

More precisely we introduce the following concept.

Definition 1.6. *Let $x \in X$, $x^* \in X^*$, $\varepsilon > 0$ and $A \in \Sigma$. A function $f \in L_1(A, \mu, X)$ is said to be an (x, x^*, ε, A)-peak if there is a subset $A_1 \subset A$ with $\mu(A_1) < \varepsilon$ such that*

1. $f(t) = x$ *for* $t \in A \setminus A_1$;
2. $\int_{A_1} \|f(t)\|\, d\mu(t) \le (1+\varepsilon)\mu(A)\|x\|$,
3. $|\int_A x^*(f(t))\, d\mu(t)| < \varepsilon$.

An operator $T\colon L_1(\mu, X) \to E$ is said to be L-narrow if for every $x \in X$, $x^* \in X^*$, $\varepsilon > 0$ and $A \in \Sigma$ there is an (x, x^*, ε, A)-peak f with $\|T(f)\| < \varepsilon$.

The L-narrow operators form a generalization of the property which is a characterization of narrow operators in the scalar case according to Theorem 1.5.

We will prove that every L-narrow operator on $L_1(\mu, X)$ is narrow (see Th. 2.4).

In Theorem 3.5 we shall describe properties of X which are sufficient for the coincidence of the classes of L-narrow and narrow operators on $L_1(\mu, X)$. In Ex. 3.8 it will be shown, however, that for some spaces the classes mentioned above do not coincide.

2. L-Narrow Operators

First of all we prove a lemma that has also been used in [4].

Lemma 2.1. *Let $x^* \in S(X^*)$, $\varepsilon > 0$. Then for every $x \in S(x^*, \varepsilon)$ and every $\delta \in (0, \varepsilon)$ there is a $y^* \in S(X^*)$ such that $x \in S(y^*, \delta)$ and $S(y^*, \delta) \subset S(x^*, \varepsilon)$.*

P r o o f. Fix a supporting functional f_x of x. Let $\alpha_0 > 0$ be a root of equation

$$\frac{1 + \alpha(1-\varepsilon)}{\|f_x + \alpha x^*\|} = 1 - \delta. \tag{2.2}$$

Such a root exists, because the left part $F(\alpha)$ of (2.2) is a continuous function of α, $F(0) = 1 > 1 - \delta$ and $\lim_{\alpha\to\infty} F(\alpha) = 1 - \varepsilon < 1 - \delta$. Put

$$y^* = \frac{f_x + \alpha_0 x^*}{\|f_x + \alpha_0 x^*\|}.$$

Then

$$y^*(x) = \frac{1 + \alpha_0 x^*(x)}{\|f_x + \alpha_0 x^*\|} > \frac{1 + \alpha_0(1-\varepsilon)}{\|f_x + \alpha_0 x^*\|} = 1 - \delta,$$

i.e., $x \in S(y^*, \delta)$. To prove the inclusion $S(y^*, \delta) \subset S(x^*, \varepsilon)$ we take an arbitrary $y \in S(y^*, \delta)$. Then

$$1 + \alpha_0 x^*(y) \ge f_x(y) + \alpha_0 x^*(y) > (1-\delta)\|f_x + \alpha_0 x^*\| = 1 + \alpha_0(1-\varepsilon).$$

So $x^*(y) > 1 - \varepsilon$, which means $y \in S(x^*, \varepsilon)$.

Next we formulate and prove a criterion for an operator defined on $L_1(\mu, X)$ to be narrow.

Theorem 2.2. *For an operator $T\colon L_1(\mu, X) \to E$ the following are equivalent:*

1. *$T\colon L_1(\mu, X) \to E$ is narrow.*
2. *For every $x, y \in X$, $x^* \in X^*$, $\varepsilon > 0$ and $A \in \Sigma$ there is a function $f \in L_1(A, \mu, X)$ with*

$$\begin{aligned} \Big|\int\limits_A x^*(f(t) - x)\, d\mu(t)\Big| &< \varepsilon, \\ \|T(f - x\chi_A)\| &< \varepsilon, \\ \|f\| &= \mu(A)\|x\|, \\ \|f + y\chi_A\| &> (1-\varepsilon)\mu(A)(\|x\| + \|y\|). \end{aligned}$$

P r o o f. (1) $\Rightarrow$ (2). The cases $x = 0$ or $\mu(A) = 0$ are trivial (just take $f = 0$), so we may exclude them from our consideration. Since $L_1(\Omega, \mu, X)$ can be represented as l_1-sum of $L_1(A, \mu, X)$ and $L_1(\Omega \setminus A, \mu, X)$, the restriction of T to $L_1(A, \mu, X)$ is narrow [3, Th. 4.4]. To deduce the statement (2) let us apply Remark 1.3 to the restriction of T to $L_1(A, \mu, X)$, sufficiently small $\varepsilon_1 > 0$, the element $\hat{x} = \frac{x\chi_A}{\|x\chi_A\|} \in S(L_1(A, \mu, X))$, the weak neighbourhood W of $\hat{x}$ consisting of all functions $g \in L_1(A, \mu, X)$ with $|\int_A x^*(g(t) - \hat{x}(t))\, d\mu(t)| < \varepsilon_1$, and the element $\hat{y} = \frac{y\chi_A}{\|x\chi_A\|} \in S(L_1(A, \mu, X))$. Then we get an element $\hat{z} \in W \cap S(L_1(A, \mu, X))$ with the properties that $\|T(\hat{z} - \hat{x})\| < \varepsilon_1$ and $\|\hat{z} + \hat{y}\| > 2 - \varepsilon_1$. Then $f = \|x\|\mu(A)\hat{z}$ will be what we need.

(2) $\Rightarrow$ (1). Let $x, y \in S(L_1(\mu, X))$, and S be an ε-slice of $S(L_1(\mu, X))$ containing x. Fix $\varepsilon_1 < \varepsilon/2$ and a slice $S_1 = S(x^*, \varepsilon_1) \subset S$ such that $x \in S_1$ (we use Lemma 2.1).

By density arguments we may assume without loss of generality that x and y are step functions taking values in a finite dimensional subspace $Y \subset X$. Let $e^* = x^*|_{L_1(\mu, Y)}$. Observe that $L_1{}^*(\mu, Y) \cong L_\infty(\mu, Y^*)$, so we can consider $e^* \in S(L_\infty(\mu, Y^*))$. Then $\langle e^*, x\rangle > 1 - \varepsilon_1$ (since $x(t) \in Y$, $\langle e^*, x\rangle$ makes sense). By finite dimensionality of Y, e^* can be approximated by step functions. So we may assume that there is a partition $A_1, \dots, A_n$ of Ω such that $x = \sum_{k=1}^n x_k \chi_{A_k}$, $y = \sum_{k=1}^n y_k \chi_{A_k}$, where $x_k, y_k \in Y$, and $e^* = \sum_{k=1}^n e_k^* \chi_{A_k}$, $e_k^* \in Y^*$. Extend e_k^* to be elements of X^* of the same norm.

For every $k = 1, \dots, n$ apply condition (2) to $\delta > 0$, x_k, y_k, e_k^* and A_k. So there exist $f_k \in L_1(A_k, \mu, X)$ with

$$\begin{aligned} \|T(f_k - x_k\chi_{A_k})\| &< \delta, \\ \Big|\int\limits_{A_k} e_k^*(f_k(t) - x_k)\, d\mu(t)\Big| &< \delta, \\ \|f_k\| &= \mu(A_k)\|x\|, \\ \|f_k + y_k\chi_{Ak}\| &> (1-\delta)\mu(A_k)(\|x_k\| + \|y_k\|). \end{aligned}$$

Define $v, z \in L_1(\mu, X)$ as follows:

$$v = \sum_{k=1}^n f_k, \qquad z = \frac{v}{\|v\|_1}.$$

When δ is small enough, the element $z \in S(L_1(\mu, X))$ will satisfy all the conditions of the definition of a narrow operator.

Remark 2.3. *Let $T\colon L_1(\mu, X) \to E$ be an L-narrow operator. Then for every $x \in X$, $x^* \in X^*$, $\varepsilon > 0$ and $A \in \Sigma$ there is an (x, x^*, ε, A)-peak g with $\|T(g)\| < \varepsilon$ and with*

$$\int\limits_{A_1} \|g(t)\|\, d\mu(t) = (1+\varepsilon)\mu(A)\|x\|$$

for a corresponding $A_1 \subset A$ from Definition 1.6.

P r o o f. Let $\varepsilon < 1$, $\varepsilon_1 < \varepsilon/2$ and f be an $(x, x^*, \varepsilon_1, A)$-peak with corresponding A_1, $\mu(A_1) \neq 0$. For a $\delta < \min\{\mu(A)/2, \varepsilon\mu(A_1)/(16\mu(A))\}$ fix an (x, x^*, δ, A_1)-peak h. Consider $g_\lambda = f + \lambda h$ where $\lambda \geq 0$ is a parameter. Let us note that for $\lambda \in [0, \frac{\varepsilon}{2\delta}]$ the function g_λ is an (x, x^*, ε, A)-peak with the same A_1 as f. In fact for such a λ

$$\|T(g_\lambda)\| < \frac{\varepsilon}{2} + \lambda\delta \leq \varepsilon$$

and

$$\Big|\int\limits_A x^*(g_\lambda(t))\, d\mu(t)\Big| < \frac{\varepsilon}{2} + \lambda\delta \leq \varepsilon.$$

Consider $F(\lambda) = \int_{A_1} \|g_\lambda(t)\|\, d\mu(t)$. If $\lambda = 0$ then $F(\lambda) < (1+\varepsilon)\mu(A)\|x\|$. For $\lambda = \frac{\varepsilon}{2\delta}$

$$F(\lambda) \geq \frac{\varepsilon}{2\delta}\|h\| - 2\mu(A)\|x\| \geq \frac{\varepsilon}{\delta}\|x\|\frac{1}{4}\mu(A_1) - 2\mu(A)\|x\| > 2\mu(A)\|x\|.$$

So there is a $\lambda_0 \in [0, \frac{\varepsilon}{2\delta}]$ with $F(\lambda_0) = (1+\varepsilon)\mu(A)\|x\|$. Then $g = g_{\lambda_0}$ is the function we need.

Theorem 2.4. *Every L-narrow operator $T\colon L_1(\mu, X) \to E$ is narrow.*

P r o o f. Let $x, y \in X$, $x^* \in X^*$, $\varepsilon > 0$ and $A \in \Sigma$, let $g \in L_1(A, \mu, X)$ be an (x, x^*, δ, A)-peak with $\|T(g)\| < \delta$ for δ small enough, and let $A_1 \subset A$ be a corresponding subset from Definition 1.6. According to the previous remark we may assume that

$$\int\limits_{A_1} \|g(t)\|\, d\mu(t) = (1+\delta)\mu(A)\|x\|.$$

Consider

$$f = -\frac{1}{1+\delta} g\chi_{A_1}.$$

Then $\|f\| = \mu(A)\|x\|$ and

$$\begin{aligned} f - x\chi_A &= -x\chi_{A_1} - \frac{1}{1+\delta} g\chi_A + \Big(\frac{1}{1+\delta} g - x\Big)\chi_{A \setminus A_1} \\ &= -x\chi_{A_1} - \frac{1}{1+\delta} g - \frac{\delta}{1+\delta} x\chi_{A\setminus A_1}. \end{aligned}$$

Hence

$$\Big|\int\limits_A x^*(f(t) - x)\, d\mu(t)\Big| \le \delta\|x\| + \frac{\delta}{1+\delta} + \frac{\delta}{1+\delta}\mu(A)\|x\|.$$

By the same argument

$$\|T(f - x\chi_A)\| \le \delta\|x\|\|T\| + \frac{\delta}{1+\delta} + \frac{\delta}{1+\delta}\mu(A)\|x\|\|T\|.$$

So, when δ is small, the first three conditions of part (2), Theorem 2.2 are satisfied. The last condition follows from the fact that the support of f is of an arbitrarily small measure δ, so $\|f + y\chi_A\|$ almost equals the sum $\|f\| + \|y\chi_A\|$.

3. Reasonable Spaces

The aim of the rest of this paper is to prove the converse to Theorem 2.4 for a wide class of spaces X (containing in particular all spaces with the RNP).

Lemma 3.1. *Let $u, v \in L_1(\Omega, \Sigma, \nu)$, $\Delta \in \Sigma$, $\delta > 0$, $u(t), v(t) \in (0, 2)$ for all $t \in \Omega$. Let us assume that*

$$\int\limits_\Delta u\, d\nu \ \ge\ 2\nu(\Delta) - \delta, \tag{3.1}$$

$$\int\limits_\Delta v\, d\nu \ \le\ \delta \tag{3.2}$$

and that there are $\alpha > 0$ and $c < 2$ such that

$$\{t \in \Delta\colon v(t) < \alpha\} \subset \{t \in \Delta\colon u(t) < c\}. \tag{3.3}$$

Then

$$\nu(\Delta) \le \frac{2\delta(1+\alpha)}{\alpha(2-c)}. \tag{3.4}$$

P r o o f. Denote $\Delta_1 = \{t \in \Delta\colon v(t) < \alpha\}$, $\Delta_2 = \{t \in \Delta\colon v(t) \ge \alpha\}$. Then according to (3.2)

$$\nu(\Delta_2) \le \frac{\delta}{\alpha}.$$

Due to (3.1)

$$2\nu(\Delta) - \delta \le \int_{\Delta_1} u\,d\nu + \int_{\Delta_2} u\,d\nu \le c\nu(\Delta) + 2\frac{\delta}{\alpha}.$$

So $(2-c)\nu(\Delta) \le 2\frac{\delta(1+\alpha)}{\alpha}$, which proves (3.4).

We now introduce a geometric condition that is in a sense opposite to the Daugavet property. We recall the following notions. The *radius* of a subset $A \subset X$ at $y \in X$ is $r_y(A) = \sup\{\|a-y\|\colon a \in A\}$, and the *Chebyshev radius* of A relative to another subset $B \subset X$ is $r_B(A) = \inf\{r_y(A)\colon y \in B\}$.

Definition 3.2. *A point $x \in S(X)$ is said to be reasonable if there is a slice $S(x^*, \varepsilon)$ with $x^*(x) = 1$, and there is a $y \in S(X)$ such that $r_y(S(x^*,\varepsilon)) < 2$. The set of all reasonable points $x \in S(X)$ will be denoted by* $\mathrm{Reas}(X)$. *A Banach space X is said to be reasonable if the closed convex hull of* $\mathrm{Reas}(X)$ *contains the whole unit ball.*

In other words, $x \in S(X)$ is reasonable if $r_{S(X)}(S) < 2$ for some slice $S = S(x^*, \varepsilon)$ as above.

Evidently, every strongly exposed point of the unit ball is reasonable. Therefore every Banach space with the Radon–Nikodym property is a reasonable space in every equivalent norm, because then every closed convex bounded subset is the closed convex hull of its strongly exposed points (see, e.g., [1, Th. 5.17]). Also, every locally uniformly convex space is reasonable. But no space with the Daugavet property is reasonable. Indeed, by [7, Lemma 2.1] a Banach space X has the Daugavet property if and only if no point in $S(X)$ is reasonable; a reformulation of that lemma is that $r_{S(X)}(S) = 2$ for every slice.

There are other nonreasonable spaces; for example, if X has the Daugavet property, then the only reasonable points of $Y = X \oplus_1 \mathbb{R}$, which fails the Daugavet property, are $(0, \pm 1)$. Indeed, $(0, \pm 1)$ are obviously strongly exposed points of Y. Now let $(x, a) \in S(Y)$ with $x \neq 0$, and let (x^*, b) be a functional in $S(Y^*) =$

$S(X^* \oplus_\infty \mathbb{R})$ attaining its norm at (x, a). Then $\|x^*\| = 1$. Consider the slice $S = S((x^*, b), \varepsilon) \subset S(Y)$ and the slice $S(x^*, \varepsilon) \subset S(X)$. By the Daugavet property there is, given a point $(y, \alpha) \in S(Y)$, some $z \in S(x^*, \varepsilon)$ such that $\|y - z\| \geq \|y\| + \|z\| - \varepsilon$. Then $(z, 0) \in S$, yet

$$\|(y, \alpha) - (z, 0)\| = \|y - z\| + |\alpha| \geq \|y\| + \|z\| + |\alpha| - \varepsilon = 2 - \varepsilon.$$

Hence (x, a) is not reasonable.

There is a hierarchy of largeness conditions of slices of the unit ball. The strongest one is the Daugavet property, viz., $r_{S(X)}(S) = 2$ for every slice. A strictly weaker property is $r_S(S) = 2$ for every slice; see [4] for more on this. Still weaker is the condition that every slice has diameter 2. The following example shows that a relatively "bad" space can also be reasonable.

Example 3.3. *Although every slice of the unit sphere of c_0 is of diameter 2, every point of the unit sphere of c_0 is a reasonable point.*

P r o o f. We first present an elementary argument that every slice of $S(c_0)$ has diameter 2; see [9] for a more general statement. Let $x^* = (a_1, a_2, \dots) \in \ell_1$ with $\sum_n |a_n| = 1$ and consider the slice $S(x^*, \varepsilon)$. Pick N so that $\sum_{n=1}^N |a_n| > 1 - \varepsilon/2$ and define $x, y \in S(c_0)$ by $x_n = \operatorname{sign} a_n$ for $n < N$, $x_N = 1$, $x_n = 0$ for $n > N$ and $y_n = \operatorname{sign} a_n$ for $n < N$, $y_N = -1$, $y_n = 0$ for $n > N$. Then $x, y \in S(x^*, \varepsilon)$ and $\|x - y\| = 2$.

Now we show that every $x \in S(c_0)$ is reasonable. Pick $k \in \mathbb{N}$ such that $|x_k| = 1$, say $x_k = 1$ without loss of generality. For the k^{th} unit vectors $e_k \in S(c_0)$ and $e_k^* \in S(\ell_1)$ we have $e_k^*(x) = 1$, and for $z = (z_1, z_2, \dots) \in S(e_k^*, \varepsilon)$ it follows $z_k > 1 - \varepsilon$ so that $\|z - e_k\| \leq 1$.

The importance of reasonable points stems from the following lemma.

Lemma 3.4. *Let $x \in \operatorname{Reas}(X)$. Then for every Banach space E, every narrow operator $U\colon L_1(\mu, X) \to E$, every $\varepsilon > 0$, every $y^* \in S(X^*)$ and every $A \in \Sigma$ there is an (x, y^*, ε, A)-peak f with $\|U(f)\| < \varepsilon$.*

P r o o f. Let $U\colon L_1(\mu, X) \to E$ be a narrow operator, $y^* \in S(X^*)$. Consider an auxiliary operator $T\colon L_1(\mu, X) \to Y = E \oplus_1 \mathbb{R}$, acting as follows:

$$T(f) = \Big(Uf, \int\limits_\Omega \langle y^*, f(t)\rangle \, d\mu\Big).$$

Being a $\sim$-sum (in the sense of [8]) of a narrow operator and a functional, T is narrow by [8, Cor. 3.14].

We need to prove that for every $\varepsilon > 0$ and every $A \in \Sigma$ there is an $f \in L_1(A, \mu, X)$ with the following properties:

1. $\mu\{t \in A\colon f(t) = x\} > \mu(A) - \varepsilon$;

2. $\int\limits_{\{t\in A\colon f(t)\neq x\}} \|f(t)\|\, d\mu(t) \le \mu(A)$ and

3. $\|T(f)\| < \varepsilon$.

According to the definition of $\mathrm{Reas}(X)$, there are $x^* \in S(X^*)$, $y \in S(X)$ and $\alpha \in (0,1)$ such that $x^*(x) = 1$ and

$$r_y(S(x^*,\alpha)) = c < 2. \tag{3.5}$$

Without loss of generality one can assume $\mu(A) = 1$ (otherwise we multiply μ by an appropriate constant). Fix a $\delta > 0$ and apply Th. 2.2; hence there is a function $g \in L_1(A,\mu,X)$ with $\|g\|_1 = 1$ and

$$\int\limits_A \langle x^*, g(t)\rangle\, d\mu \;>\; 1-\delta, \tag{3.6}$$

$$\|T(g - x\chi_A)\| \;<\; \delta, \tag{3.7}$$

$$\|g - y\chi_A\| \;>\; 2-\delta. \tag{3.8}$$

Claim. Let $B = \{t \in A\colon \|g(t)\|_X < 1\}$, $D = \{t \in A\colon \|g(t)\|_X \ge 1\}$. Then

$$\int\limits_B \|g(t)\|_X\, d\mu \;<\; \frac{2\delta(1+\alpha)}{\alpha(2-c)}, \tag{3.9}$$

$$\mu(D) \;<\; \frac{2\delta(1+\alpha)}{\alpha(2-c)}. \tag{3.10}$$

P r o o f of the Claim. Since $g \in S$, due to (3.6) we have

$$\|g\|_1 - \int\limits_A \langle x^*, g(t)\rangle\, d\mu < \delta,$$

i.e.,

$$\int\limits_A \left[1 - \left\langle x^*, \frac{g(t)}{\|g(t)\|}\right\rangle\right] \|g(t)\|\, d\mu < \delta. \tag{3.11}$$

Condition (3.8) can be rewritten as

$$\int\limits_A (\|g(t)\| + 1 - \|y - g(t)\|)\, d\mu < \delta. \tag{3.12}$$

Since the expressions under the integrals in (3.11) and (3.12) are non-negative, one can pass to a smaller set:

$$\int\limits_B \Big[1 - \Big\langle x^*, \frac{g(t)}{\|g(t)\|}\Big\rangle\Big] \|g(t)\| \, d\mu < \delta, \tag{3.13}$$

and $\int_B (\|g(t)\| + 1 - \|y - g(t)\|) \, d\mu < \delta$. The last inequality means

$$\int\limits_B \|y - g(t)\| \, d\mu > \mu(B) + \int\limits_B \|g(t)\| \, d\mu - \delta. \tag{3.14}$$

By the triangle inequality

$$\begin{aligned} \int\limits_B \|y - g(t)\| \, d\mu &\le \int\limits_B \big(\big\| \|g(t)\| y - g(t)\big\| + \big\| \|g(t)\| y - y\big\|\big) \, d\mu \\ &\le \int\limits_B \|y - \frac{g(t)}{\|g(t)\|}\| \|g(t)\| \, d\mu + \mu(B) - \int\limits_B \|g(t)\| \, d\mu. \end{aligned}$$

Substituting this into (3.14) we obtain

$$\int\limits_B \Big\|y - \frac{g(t)}{\|g(t)\|}\Big\| \|g(t)\| \, d\mu > 2 \int\limits_B \|g(t)\| \, d\mu - \delta. \tag{3.15}$$

Using (3.13) and (3.15) we can apply Lemma 3.1 to

$$d\nu = \|g(t)\| \, d\mu, \ \Delta = B, \ u(t) = \Big\|y - \frac{g(t)}{\|g(t)\|}\Big\|, \ v(t) = 1 - \Big\langle x^*, \frac{g(t)}{\|g(t)\|}\Big\rangle;$$

(condition (3.5) means exactly that (3.3) is fulfilled). This gives (3.9).

Let us now turn to the proof of (3.10). As before, passing in (3.11) and (3.12) to the smaller set D we obtain the inequalities

$$\int\limits_D \Big[1 - \Big\langle x^*, \frac{g(t)}{\|g(t)\|}\Big\rangle\Big] \, d\mu \le \int\limits_D \Big[1 - \Big\langle x^*, \frac{g(t)}{\|g(t)\|}\Big\rangle\Big] \|g(t)\| \, d\mu < \delta, \tag{3.16}$$

and

$$\int\limits_D \|y - g(t)\| \, d\mu > \mu(D) + \int\limits_D \|g(t)\| \, d\mu - \delta. \tag{3.17}$$

By the triangle inequality

$$\begin{aligned} \int\limits_D \|y - g(t)\| \, d\mu &\le \int\limits_D \Big(\Big\|y - \frac{g(t)}{\|g(t)\|}\Big\| + \Big\|g(t) - \frac{g(t)}{\|g(t)\|}\Big\|\Big) \, d\mu \\ &\le \int\limits_D \Big\|y - \frac{g(t)}{\|g(t)\|}\Big\| \, d\mu + \int\limits_D \|g(t)\| \, d\mu - \mu(D) \end{aligned}$$

Substituting this into (3.17) we obtain

$$\int_D \left\| y - \frac{g(t)}{\|g(t)\|} \right\| d\mu > 2\mu(D) - \delta. \tag{3.18}$$

Using (3.16) and (3.18) we can apply Lemma 3.1 to

$$\nu = \mu,\ \Delta = D,\ u(t) = \left\| y - \frac{g(t)}{\|g(t)\|} \right\|,\ v(t) = 1 - \left\langle x^*, \frac{g(t)}{\|g(t)\|} \right\rangle.$$

This gives (3.10).

The Claim is proved.

Now we continue the proof of Lemma 3.4. Put $f = -g\chi_D + x\chi_B$. Let us prove the properties (1) to (3) formulated at the beginning of the proof for this f under the assumption that δ is small enough.

(1) $\mu\{t \in A\colon f(t) = x\} \ge \mu(B) = \mu(A) - \mu(D) > \mu(A) - \frac{2\delta(1-\alpha)}{\alpha(2-c)}$ (we have used (3.10)).

(2) $\int\limits_{\{t\in A\colon\ f(t)\neq x\}} \|f(t)\|\, d\mu(t) \le \int_D \|g(t)\|\, d\mu \le \|g\| = 1 = \mu(A)$

(3) $\|T(f)\| \le \|T(g - x\chi_A)\| + \|T\|\|g\chi_B\| + \|T\|\mu(D)$. By (3.7), (3.9) and (3.10) this means

$$\|T(f)\| \le \delta + \frac{4\delta(1-\alpha)}{\alpha(2-c)}\|T\|.$$

This completes the proof of the lemma.

Theorem 3.5. *Let X be a reasonable space. Then every narrow operator T acting from $L_1(\mu, X)$ to any other Banach space Y is L-narrow.*

P r o o f. Let us fix $y^* \in S(X^*)$ and $A \in \Sigma$, and denote by W the set of all $x \in X$ such that for every $\varepsilon > 0$ there is an (x, y^*, ε, A)-peak f with $\|Tf\| < \varepsilon$. We have to show that $W = X$. By homogeneity it is enough to check that $W \supset S(X)$.

The previous lemma shows that $\mathrm{Reas}(X) \subset W$.

Now let $x \in S(X)$ be an arbitrary element. Fix a $\delta > 0$ and find a convex combination

$$e = \sum_{k=1}^{n} a_k y_k,$$

where $y_k \in \mathrm{Reas}(X)$, δ-approximating x: $\|x - e\| < \delta$. For every $k = 1, \dots, n$ there is a $(y_k, y^*, \frac{\delta}{n}, A)$-peak g_k with $\|Tg_k\| < \delta$. Consider

$$g = \sum_{k=1}^{n} a_k g_k$$

and denote by B the set of all $t \in A$ with $g(t) = e$.

By our construction $\mu(B) > \mu(A) - \delta$, $\|Tg\| < \delta$,

$$\int\limits_{A \setminus B} \|g(t)\| \, d\mu(t) \le (1+\delta)\mu(A) + \delta,$$

and $|\int_A x^*(g(t))\, d\mu(t)| < \delta$. So, if δ is small enough, the function $f = g + (x-e)\chi_B$ will be the (x, y^*, ε, A)-peak we need.

We are now going to present an example of a narrow operator that is not L-narrow.

Definition 3.6. *Let $T \colon X \to Y$ be a linear operator. Denote by $T_L \colon L_1(\mu, X) \to L_1(\mu, Y)$ the operator defined by $(T_L f)(t) = T(f(t))$.*

Lemma 3.7. *Let the operator $T \colon X \to Y$ be narrow. Then the operator $T_L \colon L_1(\mu, X) \to L_1(\mu, Y)$ is also narrow.*

P r o o f. Since T is narrow, for every $x, y \in S(X)$ and for every weak neighbourhood $W = \{w \colon |x^\star(w-x)| < \varepsilon\}$ of x there exists $z \in W \cap S(X^\star)$ with $\|T(x-z)\| < \varepsilon$, and $\|y+z\| > 2 - \varepsilon$.

Consider $x, y \in X$, $x^\star \in S(X^\star), \varepsilon > 0, A \in \Sigma$ and use the criterion from Theorem 2.2 for T_L. We can suppose without loss of generality that $\|x\| = 1$. Let us use the above property of a narrow operator for the vectors x, $\frac{y}{\|y\|}$ and the given ε. Then we get a vector $z \in S(X)$ such that

$$|x^\star(z-x)| < \varepsilon, \quad \|T(x-z)\| < \varepsilon, \quad \left\| \frac{y}{\|y\|} + z \right\| > 2 - \varepsilon.$$

Consider the following two cases:

1) Suppose that $\|y\| \ge 1$. Then we have

$$(2-\varepsilon)\|y\| < \|y + \|y\| \cdot z\| \le \|y+z\| + \|z - \|y\| \cdot z\| = \|y+z\| + \|y\| - 1.$$

Hence

$$\|y+z\| > 1 + \|y\| - \varepsilon\|y\| > (1-\varepsilon)(1+\|y\|) = (1-\varepsilon)(\|z\| + \|y\|).$$

2) Suppose that $\|y\| < 1$. In this case we have

$$\begin{aligned} 2 - \varepsilon &< \left\| z + \frac{y}{\|y\|} \right\| \le \|z+y\| + \left\| y - \frac{y}{\|y\|} \right\| \\ &= \|z+y\| + \Big(\frac{1}{\|y\|} - 1 \Big) \|y\| = \|z+y\| + 1 - \|y\|. \end{aligned}$$

Hence

$$\|z+y\| > \|y\| + 1 - \varepsilon > (1-\varepsilon)(1+\|y\|) = (1-\varepsilon)(\|x\|+\|y\|).$$

In both cases we have $\|z+y\| > (1-\varepsilon)(\|x\|+\|y\|)$. Now let $f = z\chi_A$, $f \in L_1(A,\mu,X)$. Then for this f we have

$$\begin{aligned}
\Big|\int_A x^*(f(t)-x)\,d\mu(t)\Big| &= \mu(A)|x^*(z-x)| < \varepsilon,\\
\|T_L(f-x\chi_A)\| &= \int_A \|T(z-x)\|\,d\mu < \varepsilon,\\
\|f\| = \mu(A) &= \mu(A)\|x\|,\\
\|f+y\chi_A\| = \mu(A)\|z+y\| &> (1-\varepsilon)\mu(A)(\|x\|+\|y\|).
\end{aligned}$$

Thus the function f satisfies all the conditions of Th. 2.2, so T_L is narrow.

Example 3.8. *Let $T\colon X \to Y$, $T \neq 0$, be a narrow operator. Then the operator T_L is an example of a narrow operator which is not L-narrow.*

P r o o f. This operator is narrow by Lemma 3.7. Let us show that an operator of the form T_L cannot be L-narrow. For this we will show that there exist $x \in X$, $x^* \in X^*$, $\varepsilon > 0$, $A \in \Sigma$ so that for every (x,x^*,ε,A)-peak $\|T_L(f)\| \geq \varepsilon$.

Let us choose $A = \Omega$, $0 < \varepsilon < \min\{\frac{1}{2}\mu(\Omega), \frac{1}{4}\mu(\Omega)\|T\|\}$. We choose the element $x \in S(X)$ so that $\|T(x)\| \geq \frac{1}{2}\|T\|$, x^* is arbitrary. Let f be an (x,x^*,ε,A)-peak. Let us estimate $\|T_L(f)\|$:

$$\begin{aligned}
\|T_L(f)\| &= \int_\Omega \|T_L(f)(t)\|\,d\mu(t) = \int_\Omega \|T(f(t))\|\,d\mu(t)\\
&\geq \int_{\{t\in\Omega\colon\, f(t)=x\}} \|T(f(t))\|\,d\mu(t) = \|T(x)\|\,\mu(\{t\in\Omega\colon f(t)=x\})\\
&> \frac{1}{2}\|T\|(\mu(\Omega)-\varepsilon) > \frac{1}{4}\mu(\Omega)\|T\| > \varepsilon.
\end{aligned}$$

Thus $\|T_L(f)\| > \varepsilon$ and T_L is not L-narrow.

There is no contradiction between this example and Th. 3.5. Indeed, if there is a narrow operator on X, then X has the Daugavet property and hence is not reasonable.

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