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On Transmission Problem for Berger Plates on an Elastic Base

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A nonlinear transmission problem for a Berger plate on an elastic base is studied. The plate consists of thermoelastic and isothermal parts. The problem generates a dynamical system in a suitable Hilbert space. In the paper the existence of a compact global attractor is proved.

Key words: transmission problem, thermoelasticity, dynamical systems, attractors.

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1. Introduction

Let Ω, Ω_1 and Ω_2 be bounded open sets in $\mathbb{R}^2$ with smooth boundaries Γ_1, $\Gamma_1 \cup \Gamma_0$ and Γ_0, respectively, such that $\Omega = \Omega_1 \cup \overline{\Omega_2}$ and $\Omega_1 \cap \Omega_2 = \emptyset$. An example is when Ω_2 is completely surrounded by Ω_1. In what follows below ν denotes the outward vector on Γ_1 and Γ_0. Also we assume that Ω_2 is a star-shaped domain, i.e., the following condition holds

$$(\mathbf{x} - \mathbf{x}_0) \cdot \nu(\mathbf{x}) \geq 0 \text{ on } \Gamma_0 \text{ for some } \mathbf{x}_0 \in \mathbb{R}^2. \tag{1.1}$$

We study an asymptotic behavior of the following system:

$$\rho_1 u_{tt} + \beta_1 \Delta^2 u + \mu \Delta\theta + F_1(u,v) = 0 \qquad \text{in } \Omega_1 \times \mathbb{R}^+, \tag{1.2}$$

$$\rho_0 \theta_t - \beta_0 \Delta\theta - \mu \Delta u_t = 0 \qquad \text{in } \Omega_1 \times \mathbb{R}^+, \tag{1.3}$$

$$\rho_2 v_{tt} + \beta_2 \Delta^2 v + F_2(u,v) = 0 \qquad \text{in } \Omega_2 \times \mathbb{R}^+. \tag{1.4}$$

Boundary conditions imposed on u along Γ_1 are clamped

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1 \times \mathbb{R}^+. \tag{1.5}$$

We assume that θ satisfies Newton's law of cooling (with the coefficient $\lambda \geq 0$) through the Γ_1 and θ vanishes along Γ_0

$$\theta = 0 \text{ on } \Gamma_0 \times \mathbb{R}^+, \quad \frac{\partial \theta}{\partial \nu} + \lambda\theta = 0 \text{ on } \Gamma_1 \times \mathbb{R}^+. \tag{1.6}$$

Also we impose the following boundary conditions along Γ_0:

$$u = v, \ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu}, \ \beta_1 \Delta u = \beta_2 \Delta v, \ \beta_1 \frac{\partial \Delta u}{\partial \nu} + \mu \frac{\partial \theta}{\partial \nu} = \beta_2 \frac{\partial \Delta v}{\partial \nu} \text{ on } \Gamma_0 \times \mathbb{R}^+. \tag{1.7}$$

Real parameters ρ_i, β_i and μ are strictly positive and the relations

$$\rho_1 \geq \rho_2 \text{ and } \beta_1 \leq \beta_2 \tag{1.8}$$

hold. Nonlinearities are given by

$$F_1(u,v) = -M(||\nabla u||^2_{\Omega_1} + ||\nabla v||^2_{\Omega_2})\Delta u + a_1(\mathbf{x})u|u|^{p-1} + g_1(\mathbf{x}, u),$$
$$F_2(u,v) = -M(||\nabla u||^2_{\Omega_1} + ||\nabla v||^2_{\Omega_2})\Delta v + a_2(\mathbf{x})v|v|^{p-1} + g_2(\mathbf{x}, v),$$

where $M(s) = s^{1+\alpha}$ with $\alpha > 0$, $a_1(\mathbf{x}) \in L^\infty(\Omega_1)$ and $a_2(\mathbf{x}) \in L^\infty(\Omega_2)$. We assume that the following condition holds:

$$\text{either } a(\mathbf{x}) \geq c_0 \ \forall \mathbf{x} \in \Omega \text{ or } 2(\alpha + 2) > p + 1, \ p \geq 1.$$

Here $a = \{a_1, a_2\}$, and $c_0 > 0$ is a small number. The functions $g_1(\mathbf{x}, u)$ and $g_2(\mathbf{x}, v)$ are scalar and satisfy the growth condition for some $\varepsilon_0 > 0$ and any $\mathbf{x}_i \in \Omega_i$

$$\left|\frac{\partial}{\partial u} g_1(\mathbf{x}_1, u)\right| + \left|\frac{\partial}{\partial v} g_2(\mathbf{x}_2, v)\right| \leq C(1 + |u|^{\max\{0, p-1-\varepsilon_0\}} + |v|^{\max\{0, p-1-\varepsilon_0\}}),$$

and, for the sake of simplicity, we assume that $g_2(\mathbf{x}, 0) = 0$.

The plate equations with nonlocal nonlinearity were introduced in [2] and their asymptotic behavior was deeply studied in [4] and [5]. Different models with partial damping were considered in [3, 7] (see also the references therein). Exponential stability of linear equations (1.2)–(1.7) ($F_i = 0$) was obtained in [12]. In [11] we proved the existence of a compact global attractor for the case when $\alpha = 0$ and $a_i = g_i = 0$.

Our main result is to prove the existence of a compact global attractor (Theorem 3.1). To obtain the result we need to overcome two difficulties. The first is to show that the corresponding energy of the system is a strict Lyapunov function, here we use the observability estimate from [1]. The second is to prove asymptotic smoothness. Here the idea of the stabilizability estimates from [5] (see also [6]) is used.

2. Preliminaries

Below the equality $w = \{u, v\}$ denotes that $w(\mathbf{x}) = u(\mathbf{x})$ if $\mathbf{x} \in \Omega_1$ and $w(\mathbf{x}) = v(\mathbf{x})$ if $\mathbf{x} \in \Omega_2$. We introduce a Hilbert space H^1_D as a space of such function $\phi \in H^1(\Omega_1)$ that $\phi = 0$ on Γ_0. The space H^1_D is equipped with the following inner product:

$$(w, \phi)_{H^1_D} := \int_{\Omega_1} \beta_0 \nabla w \cdot \nabla \phi d\mathbf{x} + \int_{\Gamma_1} \beta_0 \lambda w \phi d\mathbf{x}.$$

Denote $\mathcal{H} = H^2_0(\Omega) \times L^2(\Omega) \times L^2(\Omega_1)$. This space plays the role of a phase space for the dynamical system to be introduced below. The following set, which is densely embedded in $\mathcal{H}$, is needed for the statement about strong solutions:

$$D_0 = \left\{ \begin{array}{c} w \in \left[H^2_0(\Omega) \cap \left(H^4(\Omega_1) \times H^4(\Omega_2)\right)\right] \times H^2_0(\Omega) \times \left[H^2(\Omega_1) \cap H^1_D\right] : \\ \beta_1 \Delta w_1 = \beta_2 \Delta w_2 \text{ and } \beta_1 \frac{\partial \Delta w_1}{\partial \nu} + \mu \frac{\partial \theta}{\partial \nu} = \beta_2 \frac{\partial \Delta w_2}{\partial \nu} \text{ on } \Gamma_0, \\ \frac{\partial w_5}{\partial \nu} + \lambda w_5 = 0 \text{ on } \Gamma_1 \end{array} \right\}.$$

We introduce the potential

$$\Pi(w) = \frac{1}{2(\alpha+2)} ||\nabla w||^{2(\alpha+2)}_{L^2(\Omega)} + \frac{1}{p+1} \int_\Omega a(\mathbf{x}) |w(\mathbf{x})|^{p+1} d\mathbf{x} + \int_\Omega \int_0^{w(\mathbf{x})} g(\mathbf{x}, s) ds d\mathbf{x},$$

where $a = \{a_1, a_2\}$ and $g = \{g_1, g_2\}$. We have that $\Pi'(w) = \{F_1(w), F_2(w)\}$.

Energy functional (or Lyapunov function) $\mathcal{E} : \mathcal{H} \longrightarrow \mathbb{R}$ is defined for an argument $w = (w_1, w_2, w_3, w_4, w_5)$ (here $\{w_1, w_2\} \in H^2_0(\Omega)$, $\{w_3, w_4\} \in L^2(\Omega)$ and $w_5 \in L^2(\Omega)$) as follows:

$$\mathcal{E}(w) = \frac{1}{2} \left[\int_{\Omega_1} \beta_1 |\Delta w_1|^2 + \rho_1 |w_3|^2 + \rho_0 |w_5|^2 d\mathbf{x} \right. \\ \left. + \int_{\Omega_2} \beta_2 |\Delta w_2|^2 + \rho_2 |w_4|^2 d\mathbf{x} + 2\Pi(w_1, w_2) \right]. \tag{2.1}$$

Theorem 2.1. *Next statements hold true:*

(i) For any initial $w_0 \in \mathcal{H}$ and $T > 0$ there exists a unique mild solution $w(t) \in C([0,T]; \mathcal{H})$. Moreover, it satisfies the energy equality

$$\mathcal{E}(w(T)) - \mathcal{E}(w(t)) = -\int_t^T \int_{\Omega_1} \beta_0 |\nabla w_5|^2 d\mathbf{x} d\tau - \int_t^T \int_{\Gamma_1} \beta_0 \lambda |w_5|^2 d\Gamma d\tau \tag{2.2}$$

for all $0 \le t \le T$. If one set $S(t)w_0 = w(t)$, then $(\mathcal{H}, S(t))$ is a continuous dynamical system.

(ii) If $w_0 \in D_0$, then the corresponding mild solution is strong.

We take the same definitions of mild and strong solutions as in [10, Ch. 4]. To prove this theorem we use the standard methods from the theory of semigroups of linear operators and their perturbations, see [10]. For some details for the similar model we refer to [11].

3. Main Result

Our main result is the following theorem:

Theorem 3.1. *Let* (1.1) *and* (1.8) *hold. Then* $(\mathcal{H}, S(t))$ *possesses a compact global attractor.*

To prove this theorem, we have to prove that the energy $\mathcal{E}$ is a strict Lyapunov function for $(\mathcal{H}, S(t))$ (see Sec. 4) and $(\mathcal{H}, S(t))$ is asymptotically smooth (see Sec. 5) For how to prove the existence of a compact global attractor, taking into consideration the results of Secs. 4 and 5, we refer to [5, Cor. 2.29].

4. Strict Lyapunov Function

Proposition 4.1. *If* $\mathcal{E}(S(T)U) = \mathcal{E}(U)$ *for any* $T > 0$*, then* $S(t)U = U$ *for any* $t \geq 0$*.*

In compare with [11], our model is more complicated because of the presence of the scalar nonlinearity and the assertion is stronger since, in contrast with the proposition above, Proposition 4.13 in [11] requires $\mathcal{E}(S(T)U) = \mathcal{E}(U)$ to hold for any $T \in \mathbb{R}$. To prove Proposition 4.1 we use the Carleman-type inequalities formulated in the following auxiliary lemma (see [1, Th. 3.4]):

Lemma 4.2. *Let* w *be a solution to* $w_{tt} + \Delta^2 w = f$ *in* Ω_2 *and*

$$w|_{\Gamma_0} = \frac{\partial w}{\partial \nu}|_{\Gamma_0} = \frac{\partial^2 w}{\partial \nu^2}|_{\Gamma_0} = \frac{\partial^3 w}{\partial \nu^3}|_{\Gamma_0} = 0.$$

Then there exists such $\tau_0 > 0$ *that for all* $\tau > \tau_0$ *there holds*

$$||e^{\tau\phi} w||^2_{2,\tilde{\tau}} \leq C||e^{\tau\phi} \tilde{\tau}^{-1/2} f||, \tag{4.1}$$

where

$$||e^{\tau\phi} w||^2_{2,\tilde{\tau}} := \int_0^T \int_{\Omega_2} \tilde{\tau}^4 |e^{\tau\phi} w|^2 + \tilde{\tau}^2 |\nabla(e^{\tau\phi} w)|^2 + |\partial_t (e^{\tau\phi} w)|^2 + |\Delta(e^{\tau\phi} w)|^2 d\mathbf{x}dt$$

$\tilde{\tau} = \tau g e^{\psi}$, $\psi(\mathbf{x}) = |\mathbf{x} - \overline{\mathbf{x}}|^2$ *with* $\overline{\mathbf{x}} \in \mathbb{R}^2 \backslash \overline{\Omega_2}$, $g(t) = \frac{1}{t(T-t)}$ *and*

$$\phi(t, \mathbf{x}) = g(t)(e^{\psi(\mathbf{x})} - 2e^{||\psi||_{L^\infty(\Omega_2)}}).$$

P r o o f of Proposition 4.1. Let us consider such $T > 0$ and $U_0 \in \mathcal{H}$ that $\mathcal{E}(S(T)U_0) = \mathcal{E}(U_0)$. Energy equality (2.2) implies that $\theta \equiv 0$, then equation (1.3) implies that $u_t = 0$. Equation (1.2) implies that either $u \equiv 0$ for all $t \in [0, T]$ (case 1) or

$$M(||\nabla u||^2_{\Omega_1} + ||\nabla v||^2_{\Omega_2}) \equiv M \tag{4.2}$$

does not depend on t (case 2). Both cases are considered below.

Case 1. Let us assume $u \equiv 0$. Assume also that $||\Delta v(t)||^2_{\Omega_2} + ||v_t(t)||^2_{\Omega_2} \leq r$. Then for any $t \in [0, T]$ and $\mathbf{x} \in \Omega_2$ we have

$$|F_2(0, v)|^2 \leq \left[| \, ||\nabla w||^{1+\alpha}_{\Omega_2} \Delta v| + ||a_2||_{L^\infty} |v|^{p-1}|v| + C(r)|v|\right]^2 \leq C(r) \left[|\Delta v|^2 + |v|^2\right].$$

Using the following inequality that holds for any $t \in [0, T]$ and $\mathbf{x} \in \Omega_2$:

$$|e^{\tau\phi}\Delta w|^2 \leq |\Delta(e^{\tau\phi} w)|^2 + C\tilde{\tau}^2 |\nabla(e^{\tau\phi} w)|^2 + C\tilde{\tau}^4 |e^{\tau\phi} w|^2,$$

$\tilde{\tau}^{-1} < C/\tau$, $1 \leq C\tilde{\tau}^4$ and (4.1) with $f = F_2(0, v)$, we finally get

$$||e^{\tau\phi} w||^2_{2,\tilde{\tau}} \leq \frac{C(r)}{\tau} ||e^{\tau\phi} w||^2_{2,\tilde{\tau}}.$$

Choosing τ large enough we get the conclusion that $v \equiv 0$.

Case 2. Assume that $||\nabla v||_{\Omega_2}$ does not depend on t and (4.2) takes place. In this case we consider an application of (4.1) for $w_h(t) = v(t + h) - v(t)$ with some $h > 0$, and

$$\begin{aligned} f &= F_2(u, v(t+h)) - F_2(u, v(t)) \\ &= M\Delta w_h + a_2 \left[|v(t+h)|^{p-1} v(t+h) - |v(t)|^{p-1} v(t) \right. \\ &\quad \left. + g_2(v(t+h)) - g_2(v(t))\right]. \end{aligned}$$

Using the arguments as in case 1, we obtain $w_h(t) \equiv 0$ and, hence, v does not depend on t. ∎

5. Asymptotic Smoothness

The proof of the asymptotic smoothness is based on the method of compensated compactness function suggested in [8] and developed in [5] (see also [6]).

Let $(u^1(t), v^1(t), \theta^1(t))$ and $(u^2(t), v^2(t), \theta^2(t))$ be solutions to the problem (1.2)–(1.7) and assume that for any $t > 0$ there exists $R > 0$ such that

$$\int_{\Omega_1} \rho_1 |u^i_t|^2 + \beta_1 |\Delta u^i|^2 + \rho_0 |\theta^i|^2 d\mathbf{x} + \int_{\Omega_2} \rho_2 |v^i_t|^2 + \beta_2 |\Delta v^i|^2 d\mathbf{x} \leq R^2.$$

Let $u(t) = u^1(t) - u^2(t)$, $v(t) = v^1(t) - v^2(t)$, $\theta(t) = \theta^1(t) - \theta^2(t)$. The triple $(u(t), v(t), \theta(t))$ satisfies boundary conditions (1.5)–(1.7) and the following system:

$$\begin{cases} \rho_1 u_{tt} + \beta_1 \Delta^2 u + \mu \Delta\theta = G_1, \\ \rho_0 \theta_t - \beta_0 \Delta\theta - \mu \Delta u_t = 0, \\ \rho_2 v_{tt} + \beta_2 \Delta^2 v = G_2. \end{cases}$$

with $G_1(t) = F_1(u^2, v^2) - F_1(u^1, v^1)$ and $G_2(t) = F_2(u^2, v^2) - F_2(u^1, v^1)$.

Also we denote

$$E(t) = \frac{1}{2}\int_{\Omega_1} \rho_1 |u_t|^2 + \beta_1 |\Delta u|^2 + \rho_0 |\theta|^2 d\mathbf{x} + \frac{1}{2}\int_{\Omega_2} \rho_2 |v_t|^2 + \beta_2 |\Delta v|^2 d\mathbf{x}.$$

Proposition 5.1. *Let* (1.1) *and* (1.8) *hold. There exists* $k, C > 0$ *and a functional* $R(u, v, u_t, v_t, \theta)$*, continuous on* $\mathcal{H}$*, such that if*

$$R(t) := R(u(t), v(t), u_t(t), v_t(t), \theta(t)),$$

then $|R(t)| \le CE(t)$ *and*

$$\frac{d}{dt}R(t) \le -kE(t) + C\left[\int_{\Omega_1} |\nabla\theta|^2 d\mathbf{x} + \int_{\Omega} |\left\{u, v\right\}|^2 + |\Delta_D^{-1}\left\{\rho_1 u_t, \rho_2 v_t\right\}|^2 d\mathbf{x}\right].$$

Our proof of Proposition 5.1 mostly follows the line of arguments given in [11]. We only give here the formula for R:

$$R = J_1 + \frac{\eta}{\beta_1,} J_2 + \left(\frac{\mu}{2} - \eta C\right) J_3 + \eta^{1/2} J_4$$

with sufficiently small $\eta > 0$ and J_i defined as follows:

$$J_1 = -\int_{\Omega_1} \rho_1 u_t w_1 d\mathbf{x} - \int_{\Omega_2} \rho_2 v_t w_2 d\mathbf{x},$$

$$J_2 = \int_{\Omega_1} \rho_1 u_t h \cdot \nabla u d\mathbf{x} + \int_{\Omega_2} \rho_2 v_t h \cdot \nabla v d\mathbf{x}, \quad J_3(t) = \int_{\Omega_1} \rho_1 u_t \phi u d\mathbf{x},$$

$$J_4 = \int_{\Omega_1} \rho_1 u_t \psi m \cdot \nabla u d\mathbf{x} + \int_{\Omega_2} \rho_2 v_t \psi m \cdot \nabla v d\mathbf{x}.$$

Here $\{w_1, w_2\} := \Delta_D^{-1}\{\rho_0 \phi_1 \theta, 0\}$, where Δ_D^{-1} is an inverse Laplace operator with the Dirichlet boundary conditions on Γ_1, a vector field $h = (h_1, h_2) \in [C^2(\overline{\Omega})]^2$ satisfies $h(\mathbf{x}) = -\nu(\mathbf{x})$ if $\mathbf{x} \in \Gamma_1$, $m(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$, where $\mathbf{x}_0$ is the same as in (1.1). Functions ϕ and ψ are scalar from $C^2(\overline{\Omega})$ and $\phi(\mathbf{x}) = 1$ if $\mathbf{x} \in \Omega_1 \setminus U_{4\delta}(\Gamma_0)$ and $\phi(\mathbf{x}) = 0$ if $\mathbf{x} \in U_{2\delta}(\Gamma_0) \cap \Omega_1$; $\psi(\mathbf{x}) = 1$ if $\mathbf{x} \in U_{4\delta}(\Omega_2)$ and $\psi(\mathbf{x}) = 0$ if $\Omega_1 \setminus U_{8\delta}(\Omega_2)$. Number $\delta > 0$ is chosen sufficiently small. The idea of such J_i was used by many authors (see, e.g., [3, 6, 9, 11, 12] and the references therein).

Proposition 5.1 is a key step of the proof. We get the asymptotic smoothness using the arguments from [5, Ch. 3].

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