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# On Cyclic Functions in Weighted Hardy Spaces

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Let $H^2_\sigma(\mathbb{C}_+)$, $0<\sigma<+\infty$, be a space of analytic in $\mathbb{C}_+=\{z:\mathrm{Re} z>0\}$ functions $G$ for which

$$||G||:=\sup_{-\frac{\pi}{2}<\varphi<\frac{\pi}{2}}\left\{\int_0^{+\infty}|G(re^{i\varphi})|^2e^{-2r\sigma|\sin\varphi|}dr\right\}^{1/2}<+\infty.$$

We obtain the cyclicity conditions for functions $G\in H^2_\sigma(\mathbb{C}_+)$.

*Key words*: Beurling–Lax Theorem, cyclic function, outer function, weighted Hardy space.

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## 1. Introduction

Let $H^p(\mathbb{C}_+), 1\le p<+\infty$, be the Hardy space of analytic in the half-plane $\mathbb{C}_+=\{z:\mathrm{Re} z>0\}$ functions for which

$$||f||_*=\sup_{x>0}\left\{\int_{-\infty}^{+\infty}|f(x+iy)|^pdy\right\}^{1/p}<+\infty.$$

The properties of these spaces are described in details in [1, 2], where it is shown, in particular, that the spaces $H^p(\mathbb{C}_+)$ are Banach relative to the above norm.

The problem of completeness in $H^2(\mathbb{C}_+)$ of the system

$$\{G(z)e^{\tau z}:\tau\le 0\}, \tag{1}$$

where $G\in H^2(\mathbb{C}_+)$, was studied by P. Lax [3] (the close result for a circle was obtained by Beurling [4]). We can formulate this statement in the next form (see

[5, p. 284]). A function $G \in H^2(\mathbb{C}_+)$ is called cyclic in $H^2(\mathbb{C}_+)$ if the system (1) is complete in $H^2(\mathbb{C}_+)$.

**The Beurling–Lax Theorem.** *Let $G \in H^2(\mathbb{C}_+)$, $G \not\equiv 0$. Then the following conditions are equivalent:*

1) *$G$ is cyclic in $H^2(\mathbb{C}_+)$;*

2) *the equation*

$$\int_{-\infty}^{0} f(u+\tau)g(u)dw = 0, \quad \tau \le 0, \; g \in L^2(-\infty;0),$$

*where*

$$G(z) = \frac{1}{i\sqrt{2\pi}} \int_{-\infty}^{0} g(u)e^{uz}du,$$

*has only the trivial solution in $L^2(-\infty;0)$;*

3)*the system $\{g(u-\tau) : \tau \le 0\}$, where $g(u) = 0, u > 0$, is complete in $L^2(-\infty;0)$;*

4) *$G$ has no zero in $\mathbb{C}_+$,*

$$\overline{\lim_{x\to+\infty}} \frac{\ln|G(x)|}{x} = 0,$$

*and the singular boundary function of $G$ is constant;*

5) *$G$ is outer for $H^2(\mathbb{C}_+)$.*

T. Srinivasan and J.-K. Wang [6] generalized this result that follows from 1) ⇔ 4) ⇔ 5) for arbitrary $p \in [1;+\infty)$. The circle of ideas in the theory of the vector-valued functions and operator theory clustering around the Beurling–Lax theorem is considered in the books [7, 8].

A function $G$ is said to be outer for $H^p(\mathbb{C}_+)$ if there is the representation

$$G(z) = e^{i\alpha} \exp\left\{ \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{tz+i}{(t+iz)(1+t^2)} \ln|G(it)|dt \right\}, \quad \alpha \in \mathbb{R}, \; G \in L^p(\partial\mathbb{C}_+).$$

The singular boundary function $h$ of $G \in H^p(\mathbb{C}_+)$ is defined with accuracy to an additive constant and to the values in the points of continuity by the equality

$$h(t_2) - h(t_1) = \lim_{x\to 0+} \int_{t_1}^{t_2} \ln|G(x+iy)|dy - \int_{t_1}^{t_2} \ln|G(iy)|dy. \tag{2}$$

The generalization of this theorem for a weighted Hardy space is trivial if the weight is the module of analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}}_+$ functions $\varphi$ for which $|\varphi(z)| \leq 1, z \in \mathbb{C}_+$ (therefore, we could not find the formulation of this result). The full analog of the Beurling–Lax theorem is not found for any nontrivial weighted Hardy space.

The aim of this paper is to prove such an analog in terms of complete measure in the sense of A.F. Grishyn (see [9]).

## 2. Generalizations of Hardy Spaces

P. Rooney, J. Benedetto, H. Heinig and other authors (see, for example, [10, 11]) studied the spaces of the functions analytic in $\mathbb{C}_+$ for which

$$||f|| := \left( \int_{-\infty}^{+\infty} |w(x+iy)f(x+iy)|^p dy \right)^{1/p} < +\infty,$$

and the weight $w$ satisfies some additional conditions. They adapted many classical results, but not the Beurling–Lax theorem. The result obtained by A. Sedletskii [12] opened other way. He showed that the space $H^p(\mathbb{C}_+)$ can be defined as a class of analytical in $\mathbb{C}_+$ functions for which

$$||f|| := \sup_{-\frac{\pi}{2}<\varphi<\frac{\pi}{2}} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p dr \right\}^{1/p} < +\infty.$$

Also, the last norm is equivalent to the norm $||\cdot||_*$. Therefore, B. Vinnitskii [13] considered the following generalization of the Hardy space. Let $H^p_\sigma(\mathbb{C}_+)$, $\sigma \geq 0$, $1 \leq p < +\infty$, be the space of functions analytic in $\mathbb{C}_+$ for which

$$||f|| := \sup_{-\frac{\pi}{2}<\varphi<\frac{\pi}{2}} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|} dr \right\}^{1/p} < +\infty. \tag{3}$$

The Wiener class of the entire functions of exponential type $\leq \sigma$, which belong to $L^2(\mathbb{R})$, is the subset of $H^p_\sigma(\mathbb{C}_+)$ [14, 15]. The space $H^p_\sigma(\mathbb{C}_+)$ was studied in [13, 16], where the functions $f$ from these spaces are shown to have almost everywhere (a.e.) on $\partial\mathbb{C}_+$ the angular boundary values, which we also denote by $f(iy)$, and $f(iy)e^{-\sigma|y|} \in L^p(\mathbb{R})$. The singular boundary function of the functions $G \in H^p_\sigma(\mathbb{C}_+)$ exists [9, 17] and it is defined with accuracy to an additive constant and to the values in the points of continuity by equality (2). Thus, the space $H^p_\sigma(\mathbb{C}_+), 1 \leq p < +\infty$, is a Banach space.

## 3. The Main Result

For the formulation of the main result we will consider some spaces. By definition, put $D_{\alpha,\beta} = \{z : |\text{Re}z| < 0, \alpha < \text{Im}z < \beta\}$, $D^*_{\alpha,\beta} = \mathbb{C}\backslash\overline{D}_{\alpha,\beta}, \quad \alpha < \beta$. Let $E^p[D_{\alpha,\beta}]$ and $E^p_*[D_{\alpha,\beta}], 1 \le p < +\infty$, be the spaces of the functions $f$ analytic in $D_{\alpha,\beta}$ and $D^*_{\alpha,\beta}$, respectively, for which

$$\sup \left\{ \int\limits_{\gamma} |f(z)|^p |dz| \right\}^{1/p} < +\infty,$$

where the supremum is considered on all segments $\gamma$ which lay accordingly in $D_{\alpha,\beta}$ and $D^*_{\alpha,\beta}$ and are parallel to one of the legs of $\partial D_{\alpha,\beta}$. The functions $f$ from these spaces have [13] a.e. on $\partial D_\sigma$ the angular boundary values, which we denote $f(z)$ and $f \in L^p[\partial D_\sigma]$. Also, we suppose $D_\sigma = D_{-\sigma,\sigma}$, $D^*_\sigma = D^*_{-\sigma,\sigma}$, $E^p[D_\sigma] = E^p[D_{-\sigma,\sigma}]$, and $E^p_*[D_\sigma] = E^p_*[D_{-\sigma,\sigma}]$.

Between the spaces $H^2_\sigma(\mathbb{C}_+)$ and $E^2_*[D_\sigma]$ there exists the bijection [13] which is determined by each of the formulas

$$G(z) = \frac{1}{i\sqrt{2\pi}} \int\limits_{\partial D_\sigma} g(w) e^{zw} dw \tag{4}$$

and

$$g(w) = \frac{1}{\sqrt{2\pi}} \int\limits_0^{+\infty} G(x) e^{-xw} dx, \quad \text{Re}w > 0. \tag{5}$$

A function $G \in H^2_\sigma(\mathbb{C}_+)$ is called cyclic in $H^2_\sigma(\mathbb{C}_+)$ if the system (1) is complete in this space.

**Theorem 1.** *Let $G \in H^2_\sigma(\mathbb{C}_+), \sigma > 0, G \not\equiv 0$. Then the following conditions are equivalent:*
1) *$G$ is cyclic for $H^2_\sigma(\mathbb{C}_+)$;*
2) *the equation*

$$\int\limits_{\partial D_\sigma} f(w+\tau) g(w) dw = 0, \quad \tau \le 0,\ g \in E^2_*[D_\sigma], \tag{6}$$

*where $g$ is defined by (5), has only the trivial solution $f \in E^2[D_\sigma]$;*
3) *the system $\{g(w-\tau) : \tau \le 0\}$ is complete in $E^2_*[D_\sigma]$;*
4) *$G$ has no zero in $\mathbb{C}_+$, the singular boundary function of $G$ is constant, and*

*one of the following equivalent conditions is satisfied:*

$$a)\ \varliminf_{r\to+\infty}\left(K_G(r)-\frac{\sigma}{\pi}\ln r\right)=-\infty;$$

$$b)\ \lim_{r\to+\infty}\left(K_G(r)-\frac{\sigma}{\pi}\ln r\right)=-\infty;$$

$$c)\ G(z)\exp\left(\frac{2\sigma}{\pi}z\ln z-cz\right)\notin H^p(\mathbb{C}_+)\quad \textit{for everyone}\quad c\in\mathbb{R};$$

$$d)\ \lim_{x\to+\infty}\left(\frac{\ln|G(x)|}{x}+\frac{2\sigma}{\pi}\ln x\right)=+\infty;$$

$$e)\ \varlimsup_{x\to+\infty}\left(\frac{\ln|G(x)|}{x}+\frac{2\sigma}{\pi}\ln x\right)=+\infty,$$

$$\textit{where } K_G(r)=\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)\ln|G(it)|dt. \tag{7}$$

The Beurling–Lax theorem is not a particular case of Theorem 1 because for the case of $\sigma=0$ this theorem (but not Theorem 2) is not valid.

The equivalence of the conditions 1), 2), and 3) of Theorem 1 is established in [18]. In [19] it is shown that from condition 1) there follows 4) with condition a). In [13, 19] it is proven that if $G$ has at least one zero in $\mathbb{C}_+$ or the singular boundary function of $G$ is not constant, then $G$ is not cyclic in $H^2_\sigma(\mathbb{C}_+)$.

## 4. The Auxiliary Results

The proof of Theorem 1 is based essentially on the following two statements, the last of which may be considered as the Phragmen–Lindelof type theorem.

**Theorem 2.** *Let $G\in H^2_\sigma(\mathbb{C}_+), \sigma>0, f(z)\neq 0$, for all $z\in\mathbb{C}_+$, and a singular boundary function of the function $G$ be constant. Then the conditions* a), b), c), d), *and* e) *of Theorem 1 are equivalent.*

This result is contained in [20].

**Theorem 3.** *Suppose $\widetilde{F}_1(z)e^{-i\sigma z}\in H^2_\sigma(\mathbb{C}_+)$, $\widetilde{F}_3(z)e^{i\sigma z}\in H^2_\sigma(\mathbb{C}_+)$, $\widetilde{F}_2\in H^2_{2\sigma}(\mathbb{C}_+)$, $\widetilde{F}_2(x)e^{\frac{2\sigma}{\pi}x\ln x}\in L^2(0;+\infty)$,*

$$\widetilde{F}_1(z)+\widetilde{F}_2(z)+\widetilde{F}_3(z)\equiv 0, z\in\mathbb{C}_+, \tag{8}$$

*and*

$$\lim_{x\to+\infty}\frac{\ln|\widetilde{F}_j(x)|}{x}=-\infty,\quad j\in\{1;3\}. \tag{9}$$

*Then there exists such* $c \in \mathbb{R}$ *that*

$$\widetilde{F}_1(z)e^{-i\sigma z}e^{\frac{2\sigma}{\pi}z\ln z}e^{-cz} \in H^2(\mathbb{C}_+), \quad \widetilde{F}_3(z)e^{i\sigma z}e^{\frac{2\sigma}{\pi}z\ln z}e^{-cz} \in H^2(\mathbb{C}_+), \tag{10}$$

*where* $\ln z$ *is the main branch of the logarithm in* $\mathbb{C}_+$.

P r o o f of Theorem 3. Consider the functions

$$\widetilde{f}_j(w) = \frac{1}{\sqrt{2\pi}} \int\limits_0^{+\infty} \widetilde{F}_j(x)e^{-xw}dx, j \in \{1;2;3\}, \tag{11}$$

and suppose $D_1 = D_{-2\sigma,0}$, $D_2 = D_{-2\sigma,2\sigma}$, $D_3 = D_{0,2\sigma}$. Then from (5) we have $\widetilde{f}_j \in E_*^2[D_j], j \in \{1;2;3\}$, hence by (8) we obtain

$$\widetilde{f}_1(w) + \widetilde{f}_2(w) + \widetilde{f}_3(w) \equiv 0, \quad w \in D_2^*. \tag{12}$$

Naturally, the dual formulas

$$\widetilde{F}_j(z) = \frac{1}{i\sqrt{2\pi}} \int\limits_{\partial D_j} \widetilde{f}_j(w)e^{zw}dw, \quad j \in \{1;2;3\}, \tag{13}$$

also hold. The functions $\widetilde{f}_1, \widetilde{f}_2$ and $\widetilde{f}_3$ are entire because the integrals in the right-hand member of (11) under the condition (9) converge uniformly on every compact set on $\mathbb{C}$. Using (12) and (13), we get

$$\widetilde{F}_1(z) = -\frac{1}{i\sqrt{2\pi}} \int\limits_{\partial D_1} (\widetilde{f}_2(w) + \widetilde{f}_3(w))e^{zw}dw.$$

But $\widetilde{f}_3 \in E_*^2[D_3] \subset E^2[D_1]$, therefore $\widetilde{f}_3(w)e^{wz} \in E^1[D_1], z \in \mathbb{C}_+$ [11]

$$\int\limits_{\partial D_1} \widetilde{f}_3(w)e^{zw}dw = 0, z \in \mathbb{C}_+.$$

Thus we have

$$\widetilde{F}_1(z) = -\frac{1}{i\sqrt{2\pi}} \int\limits_{\partial D_1} \widetilde{f}_2(w)e^{zw}dw.$$

The function $\widetilde{f}_2(w)e^{zw}$ is entire for each $z \in \mathbb{C}_+$. Hence, using the Coshey theorem, in a rectangle $M_k := \{z : z \in D_1, \mathrm{Re} z > k\}, k < 0$ we obtain

$$\int\limits_{\partial M_k} \widetilde{f}_2(w)e^{zw}dw = 0.$$

Thus we have

$$\widetilde{F}_1(z) = -\frac{1}{i\sqrt{2\pi}} \int\limits_{\partial(D_1 \setminus \overline{M}_k)} \widetilde{f}_2(w) e^{zw} dw, \quad z \in \mathbb{C}_+, \, k < 0. \tag{14}$$

Furthermore,

$$|\widetilde{F}_1(x)| \le \frac{1}{\sqrt{2\pi}} \int\limits_{\partial(D_1 \setminus \overline{M}_k)} |\widetilde{f}_2(w)| e^{xu} |dw| = \frac{1}{\sqrt{2\pi}} (I_1 + I_2 + I_3),$$

where $x > 0$, $w = u + iv$, $k < 0$. Let $k = -\frac{2\sigma}{\pi} \ln x$, then by the Schwarz inequality and the formula $\widetilde{f}_2(u - 2i\sigma) \in L^2(-\infty; 0)$, for $x > 1$ we get

$$I_1 = \int\limits_{-\infty}^{k} |\widetilde{f}_2(u - 2i\sigma)| e^{xu} du \le \left( \int\limits_{-\infty}^{k} |\widetilde{f}_2(u - 2i\sigma)|^2 du \int\limits_{-\infty}^{k} e^{2xu} du \right) \frac{1}{2}$$

$$\le \left( \int\limits_{-\infty}^{0} |\widetilde{f}_2(u - 2i\sigma)|^2 du \frac{\exp\left(-\frac{4\sigma}{\pi} x \ln x\right)}{2x} \right) \frac{1}{2} = \frac{c_1}{\sqrt{x}} \exp\left( -\frac{2\sigma}{\pi} x \ln x \right).$$

We also have $\widetilde{f}_1 \in L^2(-\infty; 0)$ and $\widetilde{f}_3 \in L^2(-\infty; 0)$. If we combine this with (12), we get $\widetilde{f}_2 \in L^2(-\infty; 0)$. Analogously,

$$I_3 = \int\limits_{-\infty}^{k} |\widetilde{f}_2(u)| e^{xu} du \le \frac{c_2}{\sqrt{x}} \exp\left( -\frac{2\sigma}{\pi} x \ln x \right).$$

Further,

$$I_2 = \int\limits_{-2\sigma}^{0} |\widetilde{f}_2(k + iv)| e^{xk} dv = \exp\left( -\frac{2\sigma}{\pi} x \ln x \right) \int\limits_{-2\sigma}^{0} \left| \frac{1}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} \widetilde{F}_2(t) e^{-t(k+iv)} dt \right| dv$$

$$\le \exp\left( -\frac{2\sigma}{\pi} x \ln x \right) \int\limits_{-2\sigma}^{0} \frac{1}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} \left| \widetilde{F}_2(t) e^{-tk} \right| dt dv$$

$$= \frac{\exp\left(-\frac{2\sigma}{\pi} x \ln x\right)}{\sqrt{2\pi}} \int\limits_{-2\sigma}^{0} \int\limits_{0}^{+\infty} \left| \widetilde{F}_2(t) e^{-tk} \right| dt dv$$

$$\le \frac{\exp\left(-\frac{2\sigma}{\pi} x \ln x\right)}{\sqrt{2\pi}} 2\sigma \left( \int\limits_{0}^{+\infty} \left| \widetilde{F}_2(t) e^{\frac{2\sigma}{\pi} t \ln t} \right| 2 dt \cdot \int\limits_{0}^{+\infty} e^{\frac{-4\sigma}{\pi} t \ln t + \frac{4\sigma}{\pi} t \ln x} dt \right) \frac{1}{2}$$

$$\leq c_3 \exp\left(-\frac{2\sigma}{\pi} x \ln x\right) \left(\sqrt{x} e^{\frac{4\sigma}{\pi e} x}\right) \frac{1}{2} = c_3 \exp\left(-\frac{2\sigma}{\pi} x \ln x\right) \sqrt[4]{x} e^{\frac{4\sigma}{\pi e} x}.$$

The penultimate inequality follows from the estimation of the Laplace transform in [21, p. 326]. Therefore,

$$|\widetilde{F}_1(x)| \leq c_4 e^{c_5 x} \exp\left\{-\frac{2\sigma}{\pi} x \ln x\right\}, x > 1. \tag{15}$$

We can apply a theorem of the Phragmen–Lindelof type (see [16, 9]) to the function $\varphi_1(z) = \widetilde{F}_1(z) \exp\left\{-\frac{2\sigma}{\pi} z \ln z\right\} e^{-i\sigma z} e^{-c_5 z}$. In fact, from (15) we have $\varphi_1(x) e^{-\varepsilon x} \in L^2(0; +\infty)$, $\varepsilon > 0$, and under the condition of the theorem, $\widetilde{F}_1(z) e^{-i\sigma z} \in H^2_\sigma(\mathbb{C}_+)$, for each $\gamma \in (1, 2]$ we obtain

$$(\forall \varepsilon > 0) : \sup_{|\varphi| < \frac{\pi}{2}} \left\{ \int_0^{+\infty} \left|\varphi_1(re^{i\varphi})\right|^2 \exp\{-\varepsilon r^\gamma\} \, dr \right\} < +\infty.$$

Since $\varphi_1 \in L^2[\partial\mathbb{C}_+]$, then $\varphi_1 \in H^2(\mathbb{C}_+)$. Therefore, the first formula of (10) is proved, and the second formula can be proved in a similar way. ■

## 5. Proof of Main Results

For the proof of Theorem 1 we need some auxiliary statements.

**Lemma 1.** *Let $c \in \mathbb{R}$ be such a number that $G_c(z) := e^{cz} G(z) \in H^2_\sigma(\mathbb{C}_+)$. The equation* (8) *has a nontrivial solution if and only if the equation*

$$\int_{\partial D_\sigma} f(w + \tau) g_c(w) dw = 0, \quad \tau \leq 0,$$

*where*

$$g_c(w) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} G_c(x) e^{-xw} dx, \qquad Re w > 0,$$

*has a nontrivial solution.*

Let $T^2_\sigma(\mathbb{C}_-)$ be a set of points $F = (F_1, F_2, F_3)$, where $F_1(z) e^{-i\sigma z} \in H^2(\mathbb{C}_-)$, $F_3(z) e^{i\sigma z} \in H^2(\mathbb{C}_-)$, $F_2$ is an entire function of exponential type $\leq \sigma$, and $F_1(z) + F_2(z) + F_3(z) \equiv 0$ for $z \in \mathbb{C}_- := \{\text{Re} z < 0\}$.

**Lemma 2.** *The equalities*

$$F_j(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{l_j} f(w) e^{-zw} dw, \qquad f \in E^2[D_\sigma], \qquad j \in \{1,2,3\}, \tag{16}$$

*establish a bijection of the spaces* $T^2_\sigma(\mathbb{C}_-)$ *and* $E^2[D_\sigma]$*, where* $l_1, l_3$ *and* $l_2$ *are the legs of* $\partial D_\sigma$ *(accordingly to the rays laying under and above the real axis, and the segment* $[-i\sigma; i\sigma]$*) and their orientation coincides with the positive orientation of* $D_\sigma$*.*

**Lemma 3.** *Let* $g \in E^2_*[D_\sigma]$ *and* $G(x)\ln(2+x) \in L^2(0;+\infty)$ *for* $G$*, defined by* (4)*. Then the function* $f \in E^2[D_\sigma]$ *is a solution of* (6) *if and only if one of the following conditions is valid:*
1) *values of function* $\Phi_1$*, where*

$$\Phi_j(iy) = F_j(iy)G(iy), \quad y \in \mathbb{R}, \ j \in \{1,2,3\}, \tag{17}$$

*coincide a.e. on* $\partial\mathbb{C}_+$ *with the angular boundary values of such function* $P_1$ *that* $P_1(z)e^{-i\sigma z} \in H^1_\sigma(\mathbb{C}_+)$;
2) *values of function* $\Phi_3$ *coincide a.e. on* $\partial\mathbb{C}_+$ *with the angular boundary values of such function* $P_3$ *that* $P_3(z)e^{i\sigma z} \in H^1_\sigma(\mathbb{C}_+)$;

**Lemma 4.** *For* $f \in H^p_\sigma(\mathbb{C}_+), 1 \le p < +\infty$*, there exists the singular boundary function* $h$*. It is nonincreasing and defined with exactitude to an additive constant, and by equality (2) the values in the points of continuity* $h'(t) = 0$ *for almost all* $t \in \mathbb{R}$*.*

**Lemma 5.** *If* $f \in H^p_\sigma(\mathbb{C}_+), 1 \le p < +\infty$ *and* $f \not\equiv 0$*, then*

$$f(z) = e^{ia_0 + a_1 z} \Pi^*_f(z) S^*_f(z) \exp\left\{ \frac{1}{\pi i} \int\limits_{-\infty}^{+\infty} Q(t,z) \ln|f(it)| dt \right\}, \tag{18}$$

*where* $a_0, a_1$ *are real constants,*

$$\Pi^*_f(z) = \prod_{|\lambda_n| \le 1} \frac{z-\lambda_n}{z+\overline{\lambda}_n} \prod_{|\lambda_n|>1} \frac{1 - z/\lambda_n}{1 + z/\overline{\lambda}_n} \exp\left( \frac{z}{\lambda_n} + \frac{z}{\overline{\lambda}_n} \right),$$
$$S^*_f(z) = \exp\left\{ \frac{1}{\pi i} \int\limits_{-\infty}^{+\infty} Q(t,z) dh(t) \right\}, \tag{19}$$

$(\lambda_n)$ *is a sequence of zeroes of the function* $f$ *in* $\mathbb{C}_+$*,*

$$Q(t,z) = \frac{(tz+i)^2}{(1+t^2)^2(t+iz)}.$$

*Therefore, the conditions*

$$\sum_{|\lambda_n|\le 1} Re\lambda_n < \infty,\ \ln|f(iy)| \in L^1(-1;1),\ f(iy)e^{-\sigma|y|} \in L^p(\mathbb{R}), \tag{20}$$

$$\overline{\lim_{r\to+\infty}} \left(S_f(r) + P_f(r) - K_f(r)\right) < +\infty, \tag{21}$$

*where*

$$S_f(r) = \sum_{1<|\lambda_n|\le r} \left(\frac{1}{|\lambda_n|} - \frac{|\lambda_n|}{r^2}\right)\frac{Re\lambda_n}{|\lambda_n|}, \quad P_f(r) = \frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2} - \frac{1}{r^2}\right)|dh(t)|$$

*are valid. Here $K_f(r)$ is defined by equality (7), thus all products and integrals in (18) converge absolutely and uniformly on every compact set on $\mathbb{C}_+$.*

**Lemma 6.** *Let $(\lambda_n)$ be a sequence of numbers on $\mathbb{C}_+$ for which the first condition of (20) is satisfied, and*

$$\overline{\lim_{r\to+\infty}} \left(S_f(r) - \frac{\sigma}{\pi}\ln r\right) < +\infty. \tag{22}$$

*Then the function $\Pi_f^*$ is analytic in $\mathbb{C}_+$, and*

$$\left|\Pi_f^*(z)\right| \le \exp\left(\frac{2\sigma}{\pi}x\ln r + c_6 x\right), \quad z = x + iy = re^{i\varphi} \in \mathbb{C}_+. \tag{23}$$

**Lemma 7.** *Let $h$ be a nonincreasing function on $\mathbb{R}$, and $h'(t) = 0$ for almost all $t \in \mathbb{R}$. Then if*

$$\overline{\lim_{r\to+\infty}} \left(P_f(r) - \frac{\sigma}{\pi}\ln r\right) < +\infty, \tag{24}$$

*then the function $S_f^*$ is analytic in $\mathbb{C}_+$, and*

$$\left|S_f^*(z)\right| \le \exp\left(\frac{2\sigma}{\pi}x\ln r + c_7 x\right), \quad z = x + iy = re^{i\varphi} \in \mathbb{C}_+. \tag{25}$$

Lemma 1 is contained, in [19], Lemma 2, in [18], Lemmas 3, 4, and 5, in [19], Lemma 6, in [13], and Lemma 7, in [22].

**Lemma 8.** *Let the function $G$, defined by equality* (4)*, have no zeroes in $\mathbb{C}_+$ and the singular boundary function of $G$ be constant, and let a nontrivial solution of the equations* (6) *exist. Then there is such $(F_1, F_2, F_3) \in T_\sigma^2(\mathbb{C}_-)$ that the functions $\Phi_1$ and $\Phi_3$, defined by equalities* (17)*, are the angular boundary functions on $\partial\mathbb{C}_+$ of such functions $P_1$ and $P_3$ that $P_1(z)e^{-i\sigma z}e^{-cz} \in H_\sigma^1(\mathbb{C}_+)$*

*and* $P_3(z)e^{i\sigma z}e^{-cz} \in H^1_\sigma(\mathbb{C}_+)$ *for some* $c \in \mathbb{R}$. *Moreover, the functions* $F_1$ *and* $F_3$ *are specified analytically up to the entire functions, and*

$$\begin{gathered} F_1(z)e^{-i\sigma z}\exp\left\{-\frac{2\sigma}{\pi}z\ln z - c_7 z\right\} \in H^2_\sigma(\mathbb{C}_+), \\ F_3(z)e^{i\sigma z}\exp\left\{-\frac{2\sigma}{\pi}z\ln z - c_8 z\right\} \in H^2_\sigma(\mathbb{C}_+). \end{gathered} \tag{26}$$

P r o o f. If the equation (6) has a nontrivial solution, it is possible to consider that $G(x)\ln(2+x) \in L^2(0;+\infty)$ (otherwise, by Lemma 1 we may consider the function $G(z)e^{-c_9 z}, c_9 > 0$). Then, by Lemma 3, the functions $\Phi_1$ and $\Phi_3$, defined by equalities (17), are the angular boundary functions on $\partial\mathbb{C}_+$ of such functions $P_1$ and $P_3$ that $P_1(z)e^{-i\sigma z}e^{-c_{10}z} \in H^1_\sigma(\mathbb{C}_+)$, $P_3(z)e^{i\sigma z}e^{-c_{10}z} \in H^1_\sigma(\mathbb{C}_+)$. Let

$$\Psi_j(z) = \begin{cases} F_j(z), z \in \mathbb{C}_-, \\ \dfrac{P_j(z)}{G(z)}, z \in \mathbb{C}_+, \end{cases} j \in \{1;3\}.$$

Under conditions of the lemma, $G(z) \neq 0$ for all $z \in \mathbb{C}_+$ and the singular boundary function of $G$ is constant. If we combine this with Lemma 5, for the functions $\Psi_1$, $\Psi_3$ we get

$$\begin{aligned} \Psi_1(z) &= e^{i\sigma z}e^{ia_0+a_1 z}\Pi^*_{P_1}(z)S^*_{P_1}(z) \\ &\times \exp\left\{\frac{1}{\pi i}\int_{-\infty}^{+\infty} Q(t,z)\ln\left|\Phi_1(it)e^{\sigma t}/G(it)\right|dt\right\}, \quad z \in \mathbb{C}_+. \end{aligned} \tag{27}$$

But the angular boundary values on $\partial\mathbb{C}_+$ of the functions $\Psi_1$ from $\mathbb{C}_+$ and $\mathbb{C}_-$ coincide a.e., and $F_1(z)e^{-i\sigma z} \in H^2(\mathbb{C}_-)$. Hence, $\Phi_1(it)e^{\sigma t}/G(it) = F_1(it)e^{\sigma t} \in L^2(-\infty;+\infty)$ and [1, p. 119] imply

$$\int_{-\infty}^{+\infty} \frac{|\ln|F_1(it)e^{\sigma t}||}{1+t^2}dt < +\infty. \tag{28}$$

Using

$$\frac{1}{i}Q(t,z) = \frac{1}{it-z} - \frac{it(2+t^2)}{(1+t^2)^2} - \frac{zt^2}{(1+t^2)^2},$$

we get [1, p. 119]

$$(\exists c_{11} \in \mathbb{R}) : \exp\left\{\frac{1}{\pi i}\int_{-\infty}^{+\infty} Q(t,z)\ln\left|\Phi_1(it)e^{\sigma t}/G(it)\right|dt - c_{11}z\right\} \in H^2(\mathbb{C}_+). \tag{29}$$

From the condition (21) we get

$$\overline{\lim_{r\to+\infty}}\,(S_{P_1}(r)+P_{P_1}(r)-K_{P_1}(r))<+\infty.$$

We obviously have $K_{P_1}(r)=K_{P_1(z)e^{-i\sigma z}}(r)=K_{\Psi_1(z)e^{-i\sigma z}}(r)+K_G(r)$, and after a designation $\ln^+ t=\max\{\ln t;0\}$, by (28), we obtain

$$K_{\Psi_1(z)e^{-i\sigma z}}(r)\geq\frac{-1}{2\pi}\int\limits_{1<|t|\leq r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)\ln^+\frac{1}{|\Psi_1(it)e^{\sigma t}|}dt$$

$$\geq\frac{-1}{2\pi}\int\limits_{1<|t|\leq r}\frac{1}{t^2}\ln^+\frac{1}{|\Psi_1(it)e^{\sigma t}|}dt\geq\frac{-1}{\pi}\int\limits_{1<|t|\leq r}\frac{\left|\ln\left|\Psi_1(it)e^{\sigma t}\right|\right|}{t^2+1}dt>-\infty.$$

Since

$$K_G(r)\leq\frac{1}{2\pi}\int\limits_{1<|t|\leq r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)\ln^+|G(it)|dt$$

$$\leq\frac{1}{2\pi}\int\limits_{1<|t|\leq r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)\ln^+\left|G(it)e^{-\sigma|t|}\right|dt+\frac{1}{2\pi}\int\limits_{1<|t|\leq r}\frac{1}{t^2}\sigma|t|dt$$

$$\leq\frac{1}{2\pi}\int\limits_{1<|t|\leq r}\frac{1}{t^2}\left|G(it)e^{-\sigma|t|}\right|2dt+\frac{\sigma}{2\pi}\int\limits_{1<|t|\leq r}\frac{1}{|t|}dt\leq c_{12}+\frac{\sigma}{\pi}\ln r,$$

it follows

$$\overline{\lim_{r\to+\infty}}\left(S_{P_1}(r)+P_{P_1}(r)-\frac{\sigma}{\pi}\ln r\right)<+\infty.$$

We obviously have $S_{\Psi_1}(r)=S_{P_1}(r)$, $P_{\Psi_1}(r)=P_{P_1}(r)$ and nonnegativity of $S_{\Psi_1}$ and $P_{\Psi_1}$. Then the conditions (22) and (24) are valid for the function $\Psi_1$. From this, considering the first condition of (20), from Lemmas 6 and 7, we get the estimations (23) and (25). From (23), (25) and (29) it follows that the function $\Psi_1$ belongs to the Smirnov class $E^2\subset E^1$ in $\triangle_c(0;1)$ for each $c\in\mathbb{R}$, where $\triangle_c(a;b)=\{z: a<\mathrm{Re}z<b, c<\mathrm{Im}z<c+1\}$. As $\Psi_1(z)e^{-i\sigma z}\in H^2(\mathbb{C}_-)$, it also follows that this function belongs to the class $E^2\subset E^1$ in $\triangle_c(-1;0)$ for each $c\in\mathbb{R}$. Therefore [22, Ch. 3, § 7] for $z\in\triangle_c(-1;0)\cup\triangle_c(0;1)$ the representation

$$\Psi_1(z)=\frac{1}{2\pi i}\int\limits_{\partial\triangle_c(-1;0)}\frac{\Psi_1(t)}{t-z}dt+\frac{1}{2\pi i}\int\limits_{\partial\triangle_c(0;1)}\frac{\Psi_1(t)}{t-z}dt=\frac{1}{2\pi i}\int\limits_{\partial\triangle_c(-1;1)}\frac{\Psi_1(t)}{t-z}dt$$

is valid.

As $\Psi_1\in L^2[\partial\triangle_c(-1;1)]$, [23, Ch. 3, § 5], then the function $\Psi_1$ is analytic in $\triangle_c(-1;1)$ for every $c\in\mathbb{R}$. By Lemmas 6 and 7, the functions $\Pi^*_{\Psi_1}$ and $S^*_{\Psi_1}$

are analytic in $\mathbb{C}_+$. Hence we have that $\Psi_1$ is an entire function. But then the singular boundary function of the function $\Psi_1$ is constant, and $S^*_{\Psi_1}(z) \equiv 1$, $P_{\Psi_1}(r) \equiv 0$. If we combine this with (27), (23) and (29), we get the first formula of (26). The second formula is proved in a similar way. ■

**Lemma 9.** *If* $(F_1, F_2, F_3) \in T^2_\sigma(\mathbb{C}_-)$, *the functions* $F_1$, $F_3$ *are entire, and*

$$(\exists c_{13} \in \mathbb{R}) : F_1(z)e^{-i\sigma z}e^{-c_{13}z} \in H^2(\mathbb{C}_+), (\exists c_{14} \in \mathbb{R}) : F_3(z)e^{i\sigma z}e^{-c_{14}z} \in H^2(\mathbb{C}_+),$$

*then* $(F_1, F_2, F_3) \equiv (0, 0, 0)$.

P r o o f. From the conditions of the lemma it follows that $F_1$ and $F_3$ are entire functions of exponential type. Let

$$d_j = \varlimsup_{x\to+\infty} \frac{\ln|F_1(x)|}{x}, j \in \{1; 3\},$$

then [1, p. 119] $F_1(z)e^{-i\sigma z}e^{-d_1 z} \in H^2(\mathbb{C}_+)$, $F_3(z)e^{i\sigma z}e^{-d_3 z} \in H^2(\mathbb{C}_+)$. If $d_1 \le 0$, $d_3 \le 0$, then $F_1(z)e^{-i\sigma z} \in H^2(\mathbb{C}_+)$ and, consequently, is bounded in $\mathbb{C}$, from what follows that $F_1 \equiv 0$ and, analogously, $F_3 \equiv 0$. Hence, $(F_1, F_2, F_3) \equiv (0, 0, 0.)$. If $d_1 > 0$ or $d_3 > 0$, [21, p. 47–49], then the functions $F_1$ and $F_3$ are the functions of the totally regular growth. For their indicators we have the estimations $h_{F_1}(\theta) = d_1 \cos\theta - \sigma \sin\theta$, $h_{F_3}(\theta) = d_3 \cos\theta + \sigma\sin\theta$, $h_{F_2}(\theta) \le \sigma|\sin\theta|$, $\theta \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$. This contradiction completes the proof. ■

P r o o f of Theorem 1. From the notes after the statement of Theorem 1 it follows that for its proof it is sufficient to prove the lack of nontrivial solutions of the equation (6) for the case when the function $G \in H^2_\sigma(\mathbb{C}_+)$ has no zeroes in $\mathbb{C}_+$, the singular boundary function of $G$ is constant and the condition b) is not valid. However, by contradiction we assume that nonzero solution $f \in E^2[D_\sigma]$ of the equations (6) exists. Then by Lemma 8 there exists $(F_1, F_2, F_3) \in T^2_\sigma(\mathbb{C}_+)$, for which the values of the functions $\Phi_1$ and $\Phi_3$, defined by (17), coincide a.e. on $\partial\mathbb{C}_+$ with the angular boundary values of such functions $P_1$ and $P_3$ that $P_1(z)e^{-i\sigma z}e^{-c_{15}z} \in H^1_\sigma(\mathbb{C}_+)$, $P_3(z)e^{i\sigma z}e^{-c_{16}z} \in H^1_\sigma(\mathbb{C}_+)$. Let

$$\widetilde{F}_j(z) = F_j(z)\exp\left(-\frac{2\sigma}{\pi}z\ln z\right)e^{-c_{17}z},\ j \in \{1; 2; 3\},$$

where $c_{17} = \max\{c_{15}, c_{16}, 0\}$. Then from the formulas of (26) and the definitions of $T^2_\sigma(\mathbb{C}_-)$, we obtain $\widetilde{F}_1(z)e^{-i\sigma z} \in H^2_\sigma(\mathbb{C}_+)$, $\widetilde{F}_2 \in H^2_{2\sigma}(\mathbb{C}_+)$ $\widetilde{F}_3(z)e^{i\sigma z} \in H^2_\sigma(\mathbb{C}_+)$, $\widetilde{F}_2(x)\exp\left(\frac{2\sigma}{\pi}z\ln z\right) \in L^2(0; +\infty)$ and (8). However, the condition d) is valid by Theorem 2. Subtracting this equality from the inequality (see [8])

$$\varlimsup_{x\to+\infty} \frac{\ln|P_j(x)|}{x} < +\infty,\ j \in \{1; 3\},$$

we get (9), that conditions of Theorem 3 are satisfied. Then the formulas of (10) are valid, hence by Lemma 9 we finally obtain $(F_1, F_2, F_3) \equiv (0, 0, 0)$. ∎

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