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On a Regular Hypersimplex Inscribed into the Multidimensional Cube

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It is proved the existence of a regular hypersimplex inscribed into the $(4n-1)$-dimensional cube under the vanishing condition of the resultant of some system of $4n-1$ algebraic equations with $4n-1$ unknown quantities.

Key words: multidimensional cube, regular simplex, Hadamard's matrix, circulant matrix, antipodal n-gons, homogeneous system resultant, necessary and saficient conditions.

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1. Introduction

It is well-known that there is no inscribing into the multidimensional cube, whose dimension is not equal to $4n-1$, of a regular simplex of the same dimension so, that all vertices of the last were vertices of the cube. As to dimension $4n-1$, H. Coxeter established already in 1933, the equivalence of this problem to the question of the existence of Hadamard's matrix of order $4n$ (see [1, p. 319]). We introduced notions of Hadamard's matrix of half-circulant type [2, p. 459] and antipodal n-gons inscribed into the regular $(2n-1)$-gon [3, p. 48], and proved that *the half-circulant Hadamard matrix of order* $4n$ *exists if and only if there exist antipodal* n*-gons inscribed into the regular* $(2n-1)$*-gon* (see [3, Th. 4]). The multidimensional problem about existence of a regular hypersimplex, inscribed into the $(4n-1)$-dimensional cube, reduced thereby to a plane problem on antipodal n-gon, what makes possible to use the methods of algebraic geometry for its solution. This is considered in the paper.

2. Definitions of Main Notions and its Characteristics

Hadamard's matrix H of order $4n$ (every its entry equals ± 1 and rows are pairwise orthogonal) is said to be *half-circulant* if it has the following form:

$$H = \begin{pmatrix} 1 & \cdots & 1 & \cdots \\ \vdots & A & \vdots & B \\ 1 & \cdots & -1 & \cdots \\ \vdots & B & \vdots & -A \end{pmatrix}. \tag{1}$$

Here A and B are square circulant matrices of order $2n-1$, more precisely, A is an usual circulant [4, p. 272], which we will call the right circulant, and B is the left circulant. If $a_1, a_2, \ldots, a_{2n-1}$ are entries of the first row of a right circulant A, then entries of its second and next rows are obtained by the cyclic permutation of previous row to the right: $a_{2n-1}, a_1, a_2, \ldots, a_{2n-2}$; $a_{2n-2}, a_{2n-1}, a_1, \ldots, a_{2n-3}$ and so on. The second and next rows of the left circulant B are obtained from its first row $b_1, b_2, \ldots, b_{2n-1}$ by the cyclic permutation of previous row to the left, namely: $b_2, b_3, \ldots, b_{2n-1}, b_1$; $b_3, b_4, \ldots, b_1, b_2$ and so on.

Let us consider in a complex plane the unit circle with the centre in the origin. Points z^k, $k = 0, 1, \ldots, 2n-2$, where $z = e^{\frac{2\pi i}{2n-1}}$, lie on this circle and are vertices of the regular $(2n-1)$-gon P_{2n-1}. Let P_n and P'_n be convex n-gons inscribed into P_{2n-1} so, that all its vertices are vertices of P_{2n-1}. We say that convex n-gons P_n and P'_n, inscribed into the regular $(2n-1)$-gon, are *antipodal*, if the total number of their diagonals and sides of the same length equals n for all admissible lengths. For all this, n-gon P_n is represented by the *generating polynomial* $p_n(z) = \sum_{k=0}^{2n-2} x_k z^k$, where $x_k = 1$ if the vertex of P^{2n-1} with number k belongs to P_n, and $x_k = 0$ in otherwise. Respectively, n-gon P'_n is represented by a polynomial $p'_n(z) = \sum_{k=0}^{2n-2} x'_k z^k$. Since P_n and P'_n are n-gons, their generating polynomials have exactly n coefficients x_k and x'_k equal 1.

The generating polynomial $p_n(z)$ has the property (see [3, Lem. 1])

$$|p_n|^2 = n + 2\sum_{k=1}^{n-1} d_k \cos\frac{2\pi k}{2n-1},$$

where d_k is the number of equal diagonals and sides of n-gon P_n, for which the vision angle (from the origin) equals $\varphi_k = \frac{2\pi k}{2n-1}$, $k = 1, 2, \ldots, n-1$. There is similar equality (with replacement d_k by d'_k) for the generating polynomial $p'_n(z)$. Since for antipodal n-gons P_n and P'_n by definition $d_k + d'_k = n$, $1 \le k \le n-1$, their generating polynomials satisfy relation $|p_n|^2 + |p'_n|^2 = n$ by Theorem 3 from [3].

As noted in Introduction, the existence of antipodal n-gons is the necessary and sufficient condition of existence of a half-circulant Hadamard matrix of order $4n$. In this connection, there is a natural question about analytical representation of the antipodal property of n-gons P_n and P'_n. To find the representation we assume that

$$\begin{aligned} x_0 &= \frac{1}{\sqrt{2n-1}}(y_0 + \sqrt{2}\sum_{j=1}^{n-1} y_j), \\ x_m &= \frac{1}{\sqrt{2n-1}}[y_0 + \sqrt{2}\sum_{j=1}^{n-1}(y_j \cos\frac{2\pi mj}{2n-1} + y_{2n-1-j}\sin\frac{2\pi mj}{2n-1})], \\ x_{2n-1-m} &= \frac{1}{\sqrt{2n-1}}[y_0 + \sqrt{2}\sum_{j=1}^{n-1}(y_j \cos\frac{2\pi mj}{2n-1} - y_{2n-1-j}\sin\frac{2\pi mj}{2n-1})], \end{aligned} \tag{2}$$

where $m = 1, 2, \ldots, n-1$.

Since x_0, x_m, x_{2n-1-m} equal 0 or 1, parameters $y_0, y_1, \ldots, y_{2n-2}$, by which they are represent, cannot be arbitrary. We obtain, solving linear system (2) with respect to these parameters,

$$\begin{aligned} y_0 &= \frac{1}{\sqrt{2n-1}}\sum_{i=0}^{2n-2} x_i, \\ y_j &= \sqrt{\frac{2}{2n-1}}[x_0 + \sum_{m=1}^{n-1}(x_m + x_{2n-1-m})\cos\frac{2\pi jm}{2n-1}], \\ y_{2n-1-j} &= \sqrt{\frac{2}{2n-1}}\sum_{m=1}^{n-1}(x_m - x_{2n-1-m})\sin\frac{2\pi jm}{2n-1}. \end{aligned} \tag{3}$$

This can be check of the direct substitution into system (2). Let us denote w_0, w_m and w_{2n-1-m} the right hand sides of equations of system (2) and consider following system of quadratic equations:

$$\begin{aligned} y_0 &= \frac{1}{\sqrt{2n-1}}\sum_{i=0}^{2n-2} w_i^2, \\ y_j &= \sqrt{\frac{2}{2n-1}}[w_0^2 + \sum_{m=1}^{n-1}(w_m^2 + w_{2n-1-m}^2)\cos\frac{2\pi jm}{2n-1}], \\ y_{2n-1-j} &= \sqrt{\frac{2}{2n-1}}\sum_{m=1}^{n-1}(w_m^2 - w_{2n-1-m}^2)\sin\frac{2\pi jm}{2n-1}. \end{aligned} \tag{4}$$

We find, if we solve it with respect to w_i^2, $i = 0, 1, \ldots, 2n-2$ (as the linear system!): $w_i^2 = x_i = w_i$, since coefficients of system (4) coincide with coefficients of system (3) and the right hand sides of equations of system (2) are denoted w_0, w_m, w_{2n-1-m}. This means that if parameters $y_0, y_1, \ldots, y_{2n-2}$ satisfy system (4), then $w_i^2 = w_i$ for all $i = 0, 1, 2, \ldots, 2n-2$, i.e. w_i, and that is x_i, can take only integer value 0 and 1. It follows from here that system (4) has with respect to $y_0, y_1, \ldots, y_{2n-2}$ 2^{2n-1} real-valued solutions, which are represented by form (3), where each x_i takes values 0 or 1 independently from the rest. Thus the following assertion is valid.

Lemma 1. *The coefficients of the polynomial $p(z) = \sum_{k=0}^{2n-2} x_k z^k$, which are represented by equalities (2), take only two values 0 and 1, if and only if their parameters $y_0, y_1, \ldots, y_{2n-2}$ satisfy conditions (4). All solutions of system (4) are real-valued, and their total number equals 2^{2n-1}.*

It should be pointed out that among 2^{2n-1} real solutions of system (4) there are C_{2n-1}^n combinations such, that $\sum_{i=0}^{2n-2} x_i = n$, which corresponds to convex n-gons inscribed into the regular $(2n-1)$-gon, with $y_0 = \frac{n}{\sqrt{2n-1}}$. Next, if $y_0, y_1, \ldots, y_{2n-2}$ and $y'_0, y'_1, \ldots, y'_{2n-2}$ are two such solutions of system (4), generating convex n-gons P_n and P'_n inscribed into the regular $(2n-1)$-gon P_{2n-1}, then they are antipodal if and only if the conditions

$$y_j^2 + y_{2n-1-j}^2 + {y'_j}^2 + {y'}_{2n-1-j}^2 = \frac{2n}{2n-1}, \tag{5}$$

are valid for all $j = 1, 2, \ldots, n-1$ (see [3, Lem. 3]).

Let $w = w(y) = w_0^3 + \sum_{m=1}^{n-1} (w_m^3 + w_{2n-1-m}^3)$ be a homogeneous polynomial of third degree with respect to coordinates of vector y, where w_0, w_m and w_{2n-1-m} are again the right hand sides of equations (2).

Lemma 2. *System (4) is represented in following equivalent form:*

$$y = \frac{1}{3} \nabla w, \tag{6}$$

where ∇w is a vector with coordinates $\frac{\partial w}{\partial y_i}$, $i = 0, 1, 2, \ldots, 2n-2$.

P r o o f. Since $\frac{\partial w_i^3}{\partial y_0} = \frac{3 w_i^2}{\sqrt{2n-1}}$ for all $i = 0, 1, 2, \ldots, 2n-2$, then the first equations in (6) has the form:

$$y_0 = \frac{1}{\sqrt{2n-1}} \sum_{i=0}^{2n-2} w_i^2,$$

which coincides with the first equation of system (4).

Since for $0 < j < n$ $\frac{\partial w_0^3}{\partial y_j} = 3\sqrt{\frac{2}{2n-1}}w_0^2$, $\frac{\partial w_m^3}{\partial y_j} = 3\sqrt{\frac{2}{2n-1}}w_m^2 \cos\frac{2\pi mj}{2n-1}$ and $\frac{\partial w_{2n-1-m}^3}{\partial y_j} = 3\sqrt{\frac{2}{2n-1}}w_{2n-1-m}^2 \cos\frac{2\pi mj}{2n-1}$, then every equation of the second group of equation in (6) has a following form:

$$y_j = \sqrt{\frac{2}{2n-1}}[w_0^2 + \sum_{m=1}^{n-1}(w_m^2 + w_{2n-1-m}^2)\cos\frac{2\pi mj}{2n-1}],$$

that coincides with the second equation of system (4).

Besides for $0 < j < n$ $\frac{\partial w_0^3}{\partial y_{2n-1-j}} = 0$, $\frac{\partial w_m^3}{\partial y_{2n-1-j}} = 3\sqrt{\frac{2}{2n-1}}w_m^2 \sin\frac{2\pi mj}{2n-1}$ and $\frac{\partial w_{2n-1-m}^3}{\partial y_{2n-1-j}} = -3\sqrt{\frac{2}{2n-1}}w_{2n-1-m}^2 \sin\frac{2\pi mj}{2n-1}$, then every equation of the third equation group in (6) has the following form:

$$y_{2n-1-j} = \sqrt{\frac{2}{2n-1}}\sum_{m=1}^{n-1}(w_m^2 - w_{2n-1-m}^2)\sin\frac{2\pi mj}{2n-1},$$

that coincides with third equation of system (4). This concludes the proof.

Since w is a homogeneous polynomial of third degree by definition, then by Euler's rule $\sum_{i=0}^{2n-2} y_i \frac{\partial w}{\partial y_i} = 3w$. Therefore, multiplying equations of system (6) respectively by coordinates $y_0, y_1, \ldots, y_{2n-2}$ of vector y and summing theirs termwise, we obtain $w = \sum_{i=0}^{2n-2} y_i^2$. Since for n-gon P_n inscribed into the regular $(2n-1)$-gon $\sum_{i=0}^{2n-2} x_i = n$, then it follows from (3) that $S = \sum_{i=0}^{2n-2} y_i^2 = n$, that is, $w = n$.

Indeed, we obtain, using trigonometrical formulas and so the identity (after the changing of summing order) $\frac{1}{2} + \sum_{j=1}^{n-1}\cos\frac{2\pi cj}{2n-1} \equiv 0$, which is valid for all integer $c \not\equiv 0(\bmod 2n-1)$:

$$S = \tfrac{n^2}{2n-1} + \sum_{j=1}^{n-1}(y_j^2 + y_{2n-1-j}^2) = \tfrac{n^2}{2n-1} + \tfrac{2}{2n-1}[(n-1)x_0^2 + \sum_{j=1}^{n-1}[2x_0\sum_{m=1}^{n-1}(x_m$$
$$+x_{2n-1-m})\cos\tfrac{2\pi jm}{2n-1} + \sum_{m=1}^{n-1}(x_m^2 + x_{2n-1-m}^2 + 2x_m x_{2n-1-m}\cos\tfrac{4\pi jm}{2n-1})$$
$$+2\sum_{m<s}(x_m x_s + x_{2n-1-m}x_{2n-1-s})\cos\tfrac{2\pi j(m-s)}{2n-1} + (x_m x_{2n-1-s} + x_{2n-1-m}x_s)$$
$$\times\cos\tfrac{2\pi j(m+s)}{2n-1}]] = \tfrac{n^2}{2n-1} + \tfrac{2}{2n-1}[(n-1)\sum_{i=0}^{2n-2}x_i^2 - x_0\sum_{m=1}^{n-1}(x_m + x_{2n-1-m})$$
$$-\sum_{m=1}^{n-1}x_m x_{2n-1-m} - \sum_{m<s}(x_m x_s + x_m x_{2n-1-s} + x_{2n-1-m}x_s + x_{2n-1-m}x_{2n-1-s})]$$
$$= \tfrac{n^2}{2n-1} + \tfrac{2}{2n-1}[n(n-1) - \tfrac{1}{2}(\sum_{i=0}^{2n-2}x_i)^2 + \tfrac{1}{2}\sum_{i=0}^{2n-2}x_i^2] = \tfrac{n^2}{2n-1} + \tfrac{2}{2n-1}\cdot\tfrac{n(n-1)}{2} = n.$$

The equation $w = n$ determine some hypersurface F in a affine space A^{2n-1}. If we pass to homogeneous coordinates $y_0, y_1, \ldots, y_{2n-2}, y_{2n-1}$, then equation $w - ny_{2n-1}^3 = 0$ represents hypersurface of third order in projective space P^{2n-1} (w is homogeneous polynomial of third degree by definition). It turns out that *the hypersurface F, representing by equation $w = n$, is a irredusible smooth hypersurface both in affine space A^{2n-1} and in projective space P^{2n-1}* (see [3, Th. 6)].

The above-mentioned results, obtained mostly in paper [3], allowed us to find following necessary and sufficient conditions of the existence of Hadamard's matrix of half-circulant type (see. Th. 5).

Theorem 1. *A half-circulant Hadamard matrix of order $4n$ exists if and only if system (6) has two solutions $y = \{y_0, y_1, \ldots, y_{2n-2}\}$ and $y' = \{y'_0, y'_1, \ldots, y'_{2n-2}\}$ such that $y_0 = y'_0 = \frac{n}{\sqrt{2n-1}}$ and so that the rest coordinates of vectors y and y' should satisfy antipodal conditions (5).*

The above solutions are obviously coordinates of the points of the cubic surfaces $w = n$.

We will mention one more result from algebraic geometry (see [5, p. 174]), which we need for the proof of our existence theorems for a regular hypersimplex inscribed into the $(4n - 1)$-dimensional cube.

Theorem 2. *Let*

$$f_i(x_0, \ldots, x_n) = 0 \quad (i = 1, \ldots, r) \tag{7}$$

be a system of homogeneous equations with undetermined coefficients and let

$$\bar{f}_i(x_0, \ldots, x_n) = 0 \quad (i = 1, \ldots, r) \tag{8}$$

be the system of equations, obtained from (7) under some given specialization of its coefficients. Then there exists a finite system of polynomials $d_1, \ldots, d_k$, depending on coefficients of equations (7) and possessing following characteristics:

(I) for some integer m

$$d_i x_0^m \equiv \sum_{j=1}^{r} a_{ij}(x_0, \ldots, x_n) f_j(x_0, \ldots, x_n),^*$$

where coefficients of polynomials $a_{ij}(x_0, \ldots, x_n)$ belong to the coefficient ring of system (7);

(II) necessary and sufficient condition for the existence of solution of system (8) in some algebraic extension of the coefficient field is the vanishing of polynomials d_i under a given specialization of coefficients.

*Sign $\equiv$ means that sum in the right hand side of this equality consists single summand $d_i x_0^m$ (after a reduction of similar terms).

Polynomials $d_1, d_2, \ldots, d_k$ of the theorem, are called the system of *resultants* or *resultant forms* for a system of homogeneous equations with several unknowns.

3. Existence Theorems

Let us introduce by analogy with the polynomial $w = w(y)$ another polynomial $w' = {w'_0}^3 + \sum_{m=1}^{n-1}({w'_m}^3 + {w'}^3_{2n-1-m})$, whose w'_i are given by the right hand sides of equalities (2), if coordinates of vector $y = \{y_0, y_1, \ldots, y_{2n-2}\}$ in them are replaced by coordinates of vector $y' = \{y'_0, y'_1, \ldots, y'_{2n-2}\}$. According to Theorem 1 the existence of a half-circulant Hadamard matrix of order $4n$ is equivalent to the solvability of certain equations. The equations can be represented in the form:

$$\begin{cases} W_i = \frac{\partial w}{\partial y_i} - 3y_i = 0, \qquad i = 0, 1, 2, \ldots, 2n-2, \\ W'_i = \frac{\partial w'}{\partial y'_i} - 3y'_i = 0, \qquad i = 0, 1, 2, \ldots, 2n-2, \\ W_{2n-1} = y_0 - \frac{n}{\sqrt{2n-1}} = 0, \quad W'_{2n-1} = y'_0 - \frac{n}{\sqrt{2n-1}} = 0, \\ Y_j = y_j^2 + y_{2n-1-j}^2 + {y'_j}^2 + {y'}^2_{2n-1-j} - \frac{2n}{2n-1} = 0, \\ \qquad j = 1, 2, \ldots, n-1. \end{cases} \tag{9}$$

Since $w_i(y)$ and $w'_i(y')$ are homogeneous polynomial of third degree with respect to its variables, then a homogeneous system, corresponding to (9), has the form:

$$\begin{cases} \bar{W}_i = \frac{\partial w}{\partial y_i} - 3y_i y_{2n-1} = 0, \qquad i = 0, 1, 2, \ldots, 2n-2, \\ \bar{W}'_i = \frac{\partial w'}{\partial y'_i} - 3y'_i y_{2n-1} = 0, \qquad i = 0, 1, 2, \ldots, 2n-2, \\ \bar{W}_{2n-1} = y_0 - \frac{n y_{2n-1}}{\sqrt{2n-1}} = 0, \quad \bar{W}'_{2n-1} = y'_0 - \frac{n y_{2n-1}}{\sqrt{2n-1}} = 0, \\ \bar{Y}_j = y_j^2 + y_{2n-1-j}^2 + {y'_j}^2 + {y'}^2_{2n-1-j} - \frac{2n y_{2n-1}^2}{2n-1} = 0, \\ \qquad j = 1, 2, \ldots, n-1. \end{cases} \tag{10}$$

System (10) consists homogeneous equations with respect to $4n-1$ unknowns $y_0, y_1, \ldots, y_{2n-2}, y_{2n-1}, y'_0, \ldots, y'_{2n-2}$ of degree less than 3. Therefore, one can obtain every of them from quadratic form (recpectively, linear form) of $4n-1$ variables under some specialization of its undetermined coefficients. According to Theorem 2 there exists a finite system of polynomials $d_1, d_2, \ldots, d_k$ whit respect to these coefficients, possessing by characteristics, indicated in the theorem, which are resultants of system (10).

Theorem 3. *Let $d_1, d_2, \ldots, d_k$ be a finite resultant system of homogeneous system (10). If every polynomial $d_1, d_2, \ldots, d_k$ vanishes after the substitution of corresponding coefficients of system (10), then one can inscribe a regular simplex of the same dimension into the $(4n-1)$-dimensional cube.*

P r o o f. Since all resultants of system (10) vanish, then it has nontrivial solution $\bar{y}_0, \dots, \bar{y}_{2n-1}, \bar{y}'_0, \dots, \bar{y}'_{2n-2}$ in some algebraic extension of its coefficient field. We shall prove that this solution is real-valued indeed.

Observe first of all that if $\bar{y}_{2n-1} = 0$, then it follows from the first equation of system (10) that $\frac{\partial \bar{w}}{\partial y_i} = 0$, $i = 0, 1, \dots, 2n-2$, where the bar means that the solution is substituted into a given partial derivative. Multiplying $\bar{W}_i$ by y_i and summing obtained equalities termwise, we have by Euler's rule: $3w - 3y_{2n-1}\sum_{i=0}^{2n-2} y_i^2 = 0$ or after a substitution of the solution: $3\bar{w} - 3\bar{y}_{2n-1}\sum_{i=0}^{2n-2} \bar{y}_i^2 = 0$. Since $\bar{y}_{2n-1} = 0$ by assumption, then $\bar{w} = 0$. That is, the point with coordinates $\bar{y}_0, \bar{y}_1, \dots, \bar{y}_{2n-2}, 0$ belongs to hypersurface F of projective space P^{2n-1}, representing by equation $W = w - ny_{2n-1}^3 = 0$. Since the homogeneous polynomial w does not depend on the variable y_{2n-1}, then both partial derivative $\frac{\partial W}{\partial y_i}$ and $\frac{\partial W}{\partial y_{2n-1}}$ vanish in the indicated point, i.e., the point $\bar{y}_0, \bar{y}_1, \dots, \bar{y}_{2n-2}, 0$ is a singular point of F. This is impossible, since the hypersurface F is irreducible and smoth in P^{2n-1} by the established above.

Consequently, $\bar{y}_{2n-1} \neq 0$. Thus one can assume that in all equations of system (10) we have $y_{2n-1} = 1$. But system (10) coincides at $y_{2n-1} = 1$ with system (9). Therefore solution $\bar{y}_0, \dots, \bar{y}_{2n-2}, 1, \bar{y}'_0, \dots, \bar{y}'_{2n-2}$ of system (10) is the solution of system (9). And since the first two groups of equations $W_i = 0$ and $W'_i = 0$ of system (9) coincide with system (6) up to notations, then vectors $\bar{y} = \{\bar{y}_0, \bar{y}_1, \dots, \bar{y}_{2n-2}\}$ and $\bar{y}' = \{\bar{y}'_0, \bar{y}'_1, \dots, \bar{y}'_{2n-2}\}$ are solutions of system (6). By Lemma 2 system (6) coincides with system (4), whose all solutions are real-valued by Lemma 1, that is, the original solution of system (10) is real-valued too.

It follows from last equations of system (9) that the coordinates of vectors $\bar{y}$ and $\bar{y}'$ satisfy the conditions $\bar{y}_0 = \bar{y}'_0 = \frac{n}{\sqrt{2n-1}}$ so and for any j is true: $\bar{y}_j^2 + \bar{y}_{2n-1-j}^2 + \bar{y}'^2_j + \bar{y}'^2_{2n-1-j} = \frac{2n}{2n-1}$. Consequently, vectors $\bar{y}$ and $\bar{y}'$ represent solutions of system (6), satisfying all conditions of Theorem 1. Thus, there exists a half-circulant Hadamard matrix H of order $4n$, having form (1). Removing from H its first column (with entries equals 1), we obtain matrix $\bar{H}$, whose rows are the coordinates of the vertices of a regular hypersimplex in E^{4n-1}, inscribed into the hypercube with edge 2, whose centre coincide with the origin (since rows of any Hadamard's matrix H are pairwise orthogonal, then the vision angle (from the origin) for each edge of the indicated hypersimplex is the same $\varphi = \arccos \frac{-1}{4n-1}$). This concludes the proof.

The resultant system of Theorem 3 consists a finite number of polynomials. This number can be very large, especially with increase of n. It happens because the number of equations of system (10) (which equals $5n-1$) exceeds significantly the number of unknown quantities $(4n-1)$. But, if both quantities are equal to each other, then the corresponding resultant system consists a single resultant.

More precisely, there exists such resultant form R that another resultant form, which belongs to the ideal of resultant forms of a given system of homogeneous equations, is divided by R [5, p. 185]. In connection with this, we modify system (10) to the following form:

$$\begin{cases} \bar{W}_i = \frac{\partial w}{\partial y_i} - 3y_i y_{2n-1} = 0, \qquad i = 0, 1, 2, \ldots, 2n-2, \\ \bar{W}'_i = \frac{\partial w'}{\partial y'_i} - 3y'_i y_{2n-1} = 0, \qquad i = 0, 1, 2, \ldots, 2n-2, \\ \qquad \bar{W}^4_{2n-1} + \bar{W}'^4_{2n-1} + \sum\limits_{j=1}^{n-1} \bar{Y}_j^2 = 0, \end{cases} \tag{11}$$

where it will be necessary to substitute in place of $\bar{W}_{2n-1}$, $\bar{W}'_{2n-1}$ and $\bar{Y}_j$ their expressions from (10). Then the number of equations of the modified system will equal $4n-1$, i.e., equate the number of unknowns.

Theorem 4. *Let R be resultant of system (11). If $R = 0$ after the substitution of coefficients of system (11), then one can inscribe a regular simplex of the same dimension into the $(4n-1)$-dimensional cube.*

P r o o f. It can be proved first as above that system (11) has a real-valued solution $\bar{y}_0, \ldots, \bar{y}_{2n-1}, \bar{y}'_0, \ldots, \bar{y}'_{2n-2}$ with $\bar{y}_{2n-1} = 1$. Then it follows from the third equation of system (11) that

$$\bar{W}_{2n-1} = \bar{y}_0 - \tfrac{n}{\sqrt{2n-1}} = 0, \qquad \bar{W}'_{2n-1} = \bar{y}'_0 - \tfrac{n}{\sqrt{2n-1}} = 0,$$
$$\bar{Y}_j = \bar{y}_j^2 + \bar{y}^2_{2n-1-j} + \bar{y}'^2_j + \bar{y}'^2_{2n-1-j} - \tfrac{2n}{2n-1} = 0,\ j = 1, 2, \ldots, n-1,$$

i.e., the given solution of (11) satisfies the last three equations of (10) too.

Thus, the coordinates of vectors $\bar{y} = \{\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_{2n-2}\}$ and $\bar{y}' = \{\bar{y}'_0, \bar{y}'_1, \ldots, \bar{y}'_{2n-2}\}$ satisfy the equations (6) and all conditions of Theorem 1, whence the assartion of our theorem follows. This concludes the proof.

If dimension of considered space is very large, the finding of even one resultant is a complex technical task. Therefore the following "negative" result may be more effective.

Theorem 5. *A half-circulant Hadamard matrix of order $4n$ does not exist if and only if there exists polynomials $A_i, A'_i, A_{2n-1}, A'_{2n-1}, B_j$, depending on variables $y_0, y_1, \ldots, y_{2n-2}, y'_0, \ldots, y'_{2n-2}$, and such that we have for nonhomogeneous system (9)*

$$\sum_{i=0}^{2n-2} (A_i W_i + A'_i W'_i) + A_{2n-1} W_{2n-1} + A'_{2n-1} W'_{2n-1} + \sum_{j=1}^{n-1} B_j Y_j \equiv 1. \tag{12}$$

P r o o f. If relation (12) is true, then, obviously, $W_i, W'_i, W_{2n-1}, W'_{2n-1}, Y_j$ cannot vanish simultaneously, i.e., system (9) have no solutions. Then any half-circulant Hadamard matrix of order $4n$ cannot exist by Theorem 1 too. Conversely, if such matrix does not exist, then system (9) has no solutions by Theorem 1. Consequently, according to Theorem 1 from [5, p. 178], there exist polynomials $A_i, A'_i, A_{2n-1}, A'_{2n-1}, B_j$ of variables $y_0, y_1, \ldots, y_{2n-2}, y'_0, \ldots, y'_{2n-2}$ such that relation (12) is valid for equations of nonhomogeneous system (9). This concludes the proof.

R e m a r k 1. The conditions of Th. 4 are satisfies, for example, if the number $2n-1$ is prime one. This follows from [2, Ths. 1 and 2].

R e m a r k 2. The role of the hypersurface of projective space P^{2n-1}, represented by equations $w = ny^3_{2n-1}$, in proofs of the existence of a regular hypersimplex inscribed into the $(4n-1)$-dimensional cube, is different from that of our paper [6]. Indeed, in the present paper the homogeneous equivalents of algebraic equations of Theorem 1, are considered actually in projective space P^{4n-2}, while in [6] they are considered on product of two projective spaces P^{2n-1} and $P'^{\,2n-1}$.

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