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On Sets with Extremely Big Slices

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A new characterization of the Radon–Nikodym property in terms of sizes of slices and equivalent norms is presented. A property opposite to the Radon–Nikodym property is studied in the context of 1-unconditional sums of Banach spaces.

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1. Introduction

In this note Ω stands for a set, Σ for a σ-algebra on Ω, λ for a probability measure on (Ω, Σ), and Σ_A^+ for the set of all positive Σ-measurable subsets of $A \in \Sigma$ whenever $\lambda(A) > 0$. The set of all probability densities supported in $A \in \Sigma_\Omega^+$ is denoted by Γ_A, i.e.,

$$\Gamma_A = \{\varphi \in L_1(\lambda) : \varphi \text{ is supported in } A, \varphi \ \geq 0, \text{ and } \|\varphi\| = 1\}.$$

X is a Banach space, $B(X)$ is the closed unit ball of X, $B(x, \alpha) = x + \alpha B(X)$ is the ball of radius α centered in x. A slice of a subset $C \subset X$ determined by a functional $x^* \in X^*$ and $\alpha > 0$ is the set

$$S(C, x^*, \alpha) = \{x \in C\colon x^*(x) \geq \sup_{y \in C} x^*(y) - \alpha\}. \tag{1}$$

The diameter of $C \subset X$ is denoted by $\mathrm{d}(C)$. The radius of C is defined as

$$\mathrm{r}(C) = \inf\{r\colon C \subset B(x, r) \text{ for some } x \in X\}.$$

We recall that a Banach space X is said to have the Radon–Nikodym property ($X \in \mathrm{RNP}$) if the following equivalent conditions hold:

(i) For every probability measurable space $(\Omega, \Sigma, \lambda)$ and for every X-valued measure μ on (Ω, Σ) if $\mu(A)/\lambda(A) \in B(X)$ for all $A \in \Sigma_\Omega^+$ then there is an $f \in L_1(\lambda, X)$ (actually from $L_\infty(\lambda, X)$) such that

$$\mu(A) = \int_A f(\omega) d\lambda(\omega) \qquad \forall A \in \Sigma.$$

(ii) Every bounded linear operator $T : L_1(\lambda) \to X$ with $T(\Gamma_\Omega) \subset B(X)$ is representable in the sense that there exists a function $f \in L_\infty(\lambda, X)$ such that

$$Tg = \int g f d\lambda \qquad \forall g \in L_1(\lambda).$$

(iii) Every closed convex bounded subset $C \subset X$ has slices of arbitrarily small diameter.

There are many other equivalent definitions of this property, for more details see, for instance, [1] and [2].

In Section 2, "Radon–Nikodym property and balls with big slices", we find criteria of this property in terms of equivalent norms and radiuses of slices of $B(X)$ (Theorem 2) and slices of general closed convex bounded sets (Theorem 1).

An obvious inequality

$$\mathrm{r}(C) \le \mathrm{d}(C) \le 2\mathrm{r}(C) \tag{2}$$

implies that we may formulate the Radon–Nikodym property equivalently as the property that every closed convex bounded set $C \subset X$ has slices of arbitrarily small radius. In the other words, the negation of the Radon–Nikodym property just means that the radiuses of slices of some closed convex bounded set are separated from zero. Theorem 2 shows that passing to equivalent norms in a space $X \notin$ RNP allows to obtain such a unit ball that the radiuses of all its slices are separated from zero even by $1 - \varepsilon$, where ε is arbitrarily small.

The following result is used in the proof of this theorem.

Definition 1. *We say that X has the r-diminition property ($X \in$ rDP) if for some positive $\alpha < 1$ (the parameter of rDP) every subset $C \subset X$ with $r(C) < \infty$ has a slice S satisfying*

$$r(S) < \alpha \cdot r(C).$$

Obviously, $X \in$ RNP if and only if $X \in$ rDP with all arbitrarily small positive values of parameter α. The result of our Theorem 1 is just the statement that a single value of $\alpha > 0$ is sufficient for RNP.

While in the second section the negation of the Radon–Nikodym property is considered as a property that the radiuses of slices of the unit balls (of equivalent norms) uniformly tend to the radiuses of these balls, in Sect. 3, "r-big slice property", we consider and investigate the limit case of these normed spaces.

Definition 2. *We say that X has the r-big slice property ($X \in \mathrm{rBSP}$) when every slice of $B(X)$ is of the radius 1.*

Obviously, this property is a strengthened negation of the Radon–Nikodym property. We investigate this notion in the context of 1-unconditional sums of sequences of spaces (by a space with a 1-unconditional basis). Namely, we investigate the connection between the rBSP of a 1-unconditional sum of a sequence of spaces and the rBSP of summands. It terns out that the sum has the rBSP provided that every summand also has the rBSP. Conversely, we obtain a complete characterization of such spaces with a 1-unconditional basis that the fixed summand inherits the rBSP of the whole 1-unconditional sum.

We may speak about the d-big slice property (dBSP), i.e., the property when every slice S of $B(X)$ is of the diameter $\mathrm{d}(S) = \mathrm{d}(B(X)) = 2$. Obviously, this property is stronger than the r-big slice property, due to the inequality (2). However, we still do not know, whether these properties coincide.

In the last section (Sect. 4 "d-big slice property") we prove that every space containing an isomorphic copy of c_0 can be equivalently renormed to possess the d-big slice property.

2. Radon–Nikodym property and balls with big slices

At first as announced above, we prove the following theorem:

Theorem 1. $X \in \mathrm{RNP}$ *if and only if* $X \in \mathrm{rDP}$.

In order to prove this theorem we modify the proof ([1], ch. 5) of the result that RNP (iii) implies RNP (ii). In this proof we use two lemmas.

Lemma 1 ([1], p. Lemma 5.6). *An operator $T : L_1(\lambda) \to X$ is representable if and only if for every $\varepsilon > 0$ and $A \in \Sigma_\Omega^+$ there exists a subset $B \in \Sigma_A^+$ with* $\mathrm{d}(T(\Gamma_B)) < \varepsilon$.

Lemma 2 ([1], p. Lemma 5.9). *Let S be a slice of $T\Gamma_A$, where $A \in \Sigma_\Omega^+$. Then there is a subset $B \in \Sigma_A^+$ such that $T\Gamma_B \subset S$.*

P r o o f o f T h e o r e m 1. Let X have the r-diminition property with a parameter $1-\delta$. We fix a bounded linear operator $T : L_1(\lambda) \to X$ satisfying $T\Gamma_\Omega \subset B(X)$ and show that it is representable using Lemma 1. Consider a set $A \in \Sigma_\Omega^+$ and $\varepsilon > 0$. By the r-diminition property of X, there is a slice S of $T\Gamma_A$ with $r(S) \le r(T\Gamma_A)\cdot(1-\delta)$. Applying Lemma 2, we find a subset $B \in \Sigma_A^+$ such that $T\Gamma_B \subset S$, hence, $r(T\Gamma_B) \le r(T\Gamma_A)\cdot(1-\delta)$.

Continuing in this way, we find a decreasing sequence of subsets $B_n \in \Sigma_A^+$ ($B_0 = A$, $B_1 = B$, ...) with the property that $r(T\Gamma_{B_n}) \le r(T\Gamma_{B_{n-1}})\cdot(1-\delta) \le r(T\Gamma_{B_0})\cdot(1-\delta)^n \to 0$. So, for some n the set B_n satisfies $r(T\Gamma_B) < \varepsilon$. By arbitrariness of ε, T is representable. So, $X \in$ RNP, as needed. The other direction is trivial. ■

Now we are able to prove the main result of this section.

Theorem 2. $X \notin$ RNP *if and only if for every* $\varepsilon > 0$ *there is such an equivalent norm* $p(x)$ *on* X *that every slice* S *of* $B_p(X)$ *satisfies* $r_p(S) > 1-\varepsilon$.

Lemma 3. *If* $X \notin$ rDP*, then for every* $\delta > 0$ *there is such a closed convex bounded symmetric subset* $C \subset X$ *that every slice* S *of* C *satisfies*

$$r(S) > (1-\delta)\cdot r(C) = 1-\delta.$$

P r o o f. Since the r-diminition property with any parameter is equivalent to the Radon–Nikodym property, X does not have the rDP with any parameter, arbitrarily close to 1. Therefore, we can find a bounded set $V \subset X$ such that every slice S of V is of the radius $r(S) > (1-\frac{\delta}{2})\cdot r(V)$. Besides, without loss of generality we may assume V to be closed and convex and to be of the radius which is greater than one. By the definition of radius, there is such $x \in X$ that $V \subset B(x, r(V)+\frac{\delta}{2})$. Denote $W = V - x$. Then $W \subset B(0, r(W)+\frac{\delta}{2})$ and every slice S of W satisfies $r(S) > (1-\frac{\delta}{2})\cdot r(W)$. Take

$$C = \mathrm{co}(W \bigcup -W).$$

Since $-W \subset B(0, r(W)+\frac{\delta}{2})$, C also lies in $B(0, r(W)+\frac{\delta}{2})$, so, $r(C) < r(W)+\frac{\delta}{2}$. Besides, $r(C) \ge r(W) > 1$. For every slice S of C either $S \bigcap W$ or $S \bigcap -W$ is a slice of W or $-W$ respectively. Therefore,

$$r(S) > (1-\tfrac{\delta}{2})\cdot r(W) > (1-\tfrac{\delta}{2})\cdot(r(C)-\tfrac{\delta}{2}) > r(C)\cdot(1-\tfrac{\delta}{2}-\tfrac{\delta}{2r(C)}) > r(C)\cdot(1-\delta).$$

Dividing C by $r(C)^{-1}$ gives us the desired set. ■

P r o o f o f T h e o r e m 2. Applying Lemma 3 to $X \in$ rDP and $\delta < \varepsilon^2$, take the corresponding set C. Now consider a new norm p taking as its closed unit ball B_p the closure of $C+\varepsilon B_{\|\cdot\|}$. Since $B_p \subset (1+\varepsilon)\cdot B_{\|\cdot\|}$, the radius r_p of every set $V \subset X$ in the sense of the new norm satisfies the inequality $r_p(V) \geq \frac{1}{1+\varepsilon} r_{\|\cdot\|}(V)$.

Let S be a slice of B_p. By the construction of B_p, there is an element $x \in \varepsilon B_{\|\cdot\|}$ and a slice $\tilde{S}$ of C such that $x+\tilde{S} \subset S$. So, $r_{\|\cdot\|}(S) > (1-\delta)\cdot r_{\|\cdot\|}(C) = 1-\delta$. Then $r_p(S) > \frac{1-\delta}{1+\varepsilon} > \frac{1-\varepsilon^2}{1+\varepsilon} = 1-\varepsilon$, as needed. The inverse implication is also valid, since the unit ball of a space with RNP has slices of arbitrarily small diameter. ■

R e m a r k. A natural question is the validity of Theorem 2 with diameters instead of radiuses. But the same technique does not suffice to prove the theorem. So, this question remains open. However, Theorem 1 still holds in the case of diameters and has the same proof.

3. r-Big slice property

In this section the unit sphere of a Banach space X is denoted by $S(X)$. The (open) slice of $B(X)$ determined by a functional $x^* \in S(X^*)$ and $\varepsilon > 0$ is the set

$$S(x^*, \varepsilon) = \{x \in B(X) \colon x^*(x) > 1-\varepsilon\}.$$

We will use this form of slices of $B(X)$ instead of slices of the general form (1) to simplify our arguments. Obviously, the r-big slice property in terms of open slices coincides with the same property in the case of closed ones.

In this section E stands for a Banach space with a 1-unconditional normalized Schauder basis. We can think of the elements of E as sequences with the property that

$$\|(a_1, a_2, \dots)\|_E = \|(|a_1|, |a_2|, \dots)\|_E \qquad \forall (a_j) \in E.$$

Suppose that $X_1, X_2, \dots$ are Banach spaces. Their E-sum $X = (X_1, X_2, \dots)_E$ consists of all sequences (x_j) with $x_j \in X_j$ and $(\|x_j\|) \in E$ with the norm $\|(x_j)\| = \|(\|x_j\|)\|_E$. Note that E^* can be represented by all sequences (a_j^*) such that

$$\sup_n \|(|a_1^*|, \dots, |a_n^*|, 0, 0, \dots)\|_{E^*} < \infty,$$

and X^* can be represented by all sequences (x_j^*), $x_j^* \in X_j^*$, such that

$$\|x^*\| = \sup_n \|(\|x_1^*\|, \dots, \|x_n^*\|, 0, 0, \dots)\|_{E^*} < \infty.$$

Theorem 3. *If $X_1, X_2, \dots \in$ rBSP, then their E-sum $X \in$ rBSP.*

P r o o f. Assume to the contrary that $S(x^*, \delta) \subset x + r\cdot B(X)$ for some $x \in X$, $r < 1$, and some slice $S(x^*, \delta)$. Consider such positive δ_k that $\sum_{k\geq 1} \delta_k < \delta/2$.

Since $\|x^*\| = 1$, there is such element $a = (a_k) \in S(E)_+$ that $\sum_{k\geq 1} \|x_k\| a_k > 1 - \delta/2$. Now for every k with $a_k \neq 0$ we consider a slice of $a_k \cdot B(X_k)$ of the form

$$S_k = \{y_k \in X_k \colon \|y_k\| \leq a_k,\ x_k^*(y_k) > \|x_k^*\| \cdot a_k - \delta_k\} \tag{3}$$

and put $S_k = \{0\}$ otherwise. Since every $y = (y_k) \in X$ with $y_k \in S_k$ for all k satisfies the inequalities

$$x^*(y) = \sum_{k\geq 1} x_k^*(y_k) > \sum_{k\geq 1} \|x_k^*\| a_k - \sum_{k\geq 1} \delta_k > 1 - \delta$$

$$\text{and } \|y\| = \|(\|y_1\|, \|y_2\|, \dots)\|_E \leq \|(a_1, a_2, \dots)\|_E = 1,$$

the following inclusion holds:

$$T_1 + T_2 + \ldots \subset S(x^*, \delta) - x \subset r \cdot B(X), \tag{4}$$

where T_k stands for $S_k - x_k$. Consider any $r' \in (r, 1)$ and let us show that

$$T_{k_0} \subset a_{k_0} \cdot r' \cdot B(X_{k_0}) \tag{5}$$

for at least one value of k_0 with $a_{k_0} \neq 0$. If such k_0 does not exist, take a $y = (y_1, y_2 \dots) \in T_1 + T_2 + \ldots$, such that $\|y_k\| \geq a_k \cdot r'$; then $\|(\|y_1\|, \|y_2\|, \dots)\| \geq \|(a_1 r', a_2 r', \dots)\| = r' \cdot 1$, which is a contradiction with (4). So, inclusion (5) holds. Dividing it by $a_{k_0} \neq 0$, we obtain the following inclusion of a slice of $B(X_{k_0})$:

$$\frac{1}{a_{k_0}} \cdot S_{k_0} \subset \frac{x_{k_0}}{a_{k_0}} + r' \cdot B(X_{k_0}),$$

which is a contradiction with $X_{k_0} \in$ rBSP. ■

For every $k = 1, 2, \dots$ consider a mapping $\gamma_k \colon E \to \mathbb{R}_+$ with

$$\gamma_k(a) = \|(a_1, \dots, a_{k-1}, 0, a_{k+1}, a_{k+2}, \dots)\|_E.$$

Theorem 4. *Consider a number $n \in \{1, 2, \dots\}$ and let*

$$\sup \gamma_n(S(h^*, \delta)) = 1 \tag{6}$$

for every slice $S(h^*, \delta)$ of $B(E)$. Let $X = (X_1, \dots, X_{n-1}, X_n, X_{n+1}, \dots)_E$. Then $X_1, \dots, X_{n-1}, X_{n+1}, \dots \in$ rBSP implies that $X \in$ rBSP, independently of properties of X.

P r o o f. Assume to the contrary that $S(x^*,\delta) \subset x + r\cdot B(X)$ for some $x \in X$, $r < 1$, and some slice $S(x^*,\delta)$. Take $h^* = (\|x_1^*\|, \|x_2^*\|, \dots) \in S(E^*)_+$ and such $\delta_k > 0$ that $\sum_{k\geq 1} \delta_k < \delta/2$. Applying (6) to $S(h^*,\delta/2)$, we get such $a \in B(E)$ that $h^*(a) > 1 - \delta/2$ and $\gamma_n(a) > \sqrt{r}$. By analogy with (3) for every k define a slice S_k and again deduce that

$$T_1 + T_2 + \dots \subset S(x^*,\delta) - x \subset r\cdot B(X), \tag{7}$$

where T_k stands for $S_k - x_k$. Observe that, due to positivity of $\gamma_n(a)$, the set $K = \{k\colon a_k \neq 0,\ k \neq n\}$ is nonempty. Let us show that

$$T_{k_0} \subset a_{k_0}\cdot\sqrt{r}\cdot B(X_{k_0}) \tag{8}$$

for at least one value of $k_0 \in K$. If such k_0 does not exist, take a $y = (y_1, \dots, y_{n-1}, 0, y_{n+1}, \dots) \in T_1 + T_2 + \dots$, such that $\|y_k\| \geq a_k\cdot\sqrt{r}$ $(k \neq n)$; then

$$\|y\| \geq \sqrt{r}\cdot\|(a_1, \dots, a_{n-1}, 0, a_{n+1}, \dots)\| = \sqrt{r}\cdot\gamma_n(a) > r,$$

which is a contradiction with (7). So, inclusion (8) holds. Dividing it by $a_{k_0} \neq 0$, we obtain the following inclusion of a slice of $B(X_{k_0})$:

$$\frac{1}{a_{k_0}}\cdot S_{k_0} \subset \frac{x_{k_0}}{a_{k_0}} + \sqrt{r}\cdot B(X_{k_0}),$$

which is a contradiction with $X_{k_0} \in$ rBSP. ■

Let us show that there is a space E satisfying the condition (6). Take $E = l_\infty^2$ and n from $\{1,2\}$. Obviously, every slice of $S(l_\infty^2)$ includes some points from the set $\{(1,1),(1,-1),(-1,1),(-1,-1)\}$ and γ_n maps every such point to 1. Therefore, (6) is fulfilled. So, $(X_1, X_2)_{l_\infty^2} \in$ rBSP if and only if one or both spaces $X_n \in$ rBSP.

It is easy to understand that every other 2-dimensional space with a 1-unconditional basis does not satisfy (6). So, the class of spaces which do not satisfy the condition (6) is not empty either.

The next Theorem 5 is the converse to Theorem 4.

Theorem 5. *Consider a number $n \in \{1, 2, \dots\}$. If there is a slice $S(h^*,\delta)$ of $S(E)$ with*

$$\sup \gamma_n(S(h^*,\delta)) < 1, \tag{9}$$

then $(X_1, X_2, \dots)_E \in$ rBSP implies $X_n \in$ rBSP.

P r o o f. Assume $n = 1$. In every other case the proof does not differ in essence. Assume to the contrary that $X_1 \notin \text{rBSP}$, i.e., $S(y_1^*, \delta_1) \subset y_1 + r_1 \cdot B(X_1)$ for some $y_1 \in X_1$, $r_1 < 1$, and some slice $S(y_1^*, \delta_1)$. Denote $C = \sup \gamma_1(S(h^*, \delta)) < 1$ and observe that an arbitrarily small δ may be chosen. Since $h_1 \neq 0$ (otherwise $C = 1$), we may take $\delta < \delta_1 \cdot |h_1^*| \cdot (1 - C)$. Finally, we may pass to a positive functional $h^* \in S(E)_+$ without any loss of generality. Now we are going to construct a slice $S(x^*, \delta)$ and an element $y \in X$ such that

$$S(x^*, \delta) \subset y + r \cdot B(X) \tag{10}$$

for some r. It will contradict with $X \in \text{rBSP}$, so, the theorem will be proved.

Take any functional $x^* \in X^*$ with $x_1^* = y_1^* \cdot h_1^*$ and satisfying $\|x_k^*\| = h_k^*$ for all k. We define $y = (y_1, 0, 0, \dots)$ and $r = 1 - (1 - r_1) \cdot (1 - C)$. Observe that it is enough to prove (10) for the set $S = S(x^*, \delta) \bigcap S(X)$ (a slice of $S(X)$) instead of $S(x^*, \delta)$. So, let us show for any $x \in S$ that $\|x - y\| < r$. If $x \in S$, then

$$x_1^*(x_1) = x^*(x) - \sum_{k \geq 2} x_k^*(x_k) > 1 - \delta - \sum_{k \geq 2} \|x_k^*\| \|x_k\| = \|x_1^*\| \|x_1\| - \delta.$$

Besides, $(\|x_1\|, \|x_2\|, \dots) \in S(h^*, \delta)$, therefore, $\|x_1\| \geq 1 - C$, since $\|x_1\| \geq \|x\| - \|(0, \|x_2\|, \|x_3\|, \dots)\| = 1 - \gamma_1(\|x_1\|, \|x_2\|, \dots)$. Consequently,

$$y_1^*(x_1) > \|x_1\| - \delta / h_1^* > \|x_1\| \cdot (1 - \delta_1 \cdot (1 - C) / \|x_1\|) > \|x_1\| \cdot (1 - \delta_1).$$

It means that x_1 lies in the slice $\|x_1\| \cdot S(y_1^*, \delta_1)$, hence, $\|x_1 - y_1\| \leq r_1 \cdot \|x_1\|$. So,

$$\|x - y\| \leq \|(\|x_1\| \cdot r_1, \|x_2\|, \|x_3\|, \dots)\| \tag{11}$$

for all $x \in S$. Define a convex function $g(r_1)$ as the right part of (11). Then

$$\|x - y\| \leq g(r_1) \leq g(1) \cdot r_1 + g(0) \cdot (1 - r_1) \leq r_1 + (1 - r_1) \cdot C = 1 - (1 - r_1)(1 - C) = r,$$

as needed. ■

R e m a r k. Let us call the spaces possessing an equivalent norm with the r-big slice property the erBSP-spaces (equivalent r-big slice property-spaces). We don't know whether all the spaces without the RNP are erBSP-spaces. We don't know even whether a space with a erBSP-subspace is a erBSP-space itself. What can be easily deduced from the previous results is that a space with a complemented erBSP-subspace is a erBSP-space. In fact, if $X = Y \oplus Z$, and Y is renormed to have the r-big slice property, then the l_∞-sum of Y and Z will be a space with the r-big slice property isomorphic to X.

There are some other properties of Banach spaces concerning extremely large slices of the unit ball, for example the Daugavet property ([3, 4]), which is strictly stronger than rBSP. The question, whether a 1-unconditional sum of spaces with the Daugavet property inherits this property, is solved [5] and the solution is far different from the result in the case of rBSP.

4. d-Big slice property

Recall that a Banach space X is said to have the d-big slice property ($X \in$ dBSP) if every slice of $S(X)$ is of diameter 2. In the other words, for every $\varepsilon > 0$ and every slice S of $S(X)$ there are such x and $y \in S$ that $\|x - y\| > 2 - \varepsilon$. For example, $c_0 \in$ dBSP. Obviously, d-big slice property is stronger than the r-big slice property, but we do not know, whether these properties coincide.

We remark that all theorems in the previous section hold true with the d-big slice property instead of r-big slice property, but the proofs are slightly longer.

Definition 3. *A Banach space X has the equivalent d-big slice property ($X \in$ edBSP) if X is isomorphic to a space having the d-big slice property.*

By the same reason as in the Remark at the end of the previous section if X has a complemented subspace with the (equivalent) d-big slice property, then $X \in$ edBSP. The case of noncomplemented subspaces is open with one exception:

Theorem 6. *If X has a subspace isomorphic to c_0, then $X \in$ edBSP.*

P r o o f. Without loss of generality we assume that c_0 is a subspace of X. Consider the collection $\mathbb{E}$ of all subspaces $Y \subset X$, such that $c_0 \subset Y$ and codimension of c_0 in Y is finite. Equip $\mathbb{E}$ with a filter $\mathcal{F}$, induced by the natural order: the base of $\mathcal{F}$ is formed by collections of the form $\{Y \in \mathbb{E} : Y \supset Y_0\}$, where $Y_0 \in \mathbb{E}$. Let $\mathcal{U}$ be an ultrafilter, majorating $\mathcal{F}$. For each $Y \in \mathbb{E}$ select a projection $P_Y : Y \to c_0$ with $\|P_Y\| \le 2$. Such a projection exists due to the Sobczyk theorem ([2, p. 71]). For every $x \in X$ and every $Y \in \mathbb{E}$ with $x \in Y$ denote $\|x\|_Y = \|P_Y x\| \bigvee \|x - P_Y x\|$, where $\bigvee$ stands for maximum of two numbers. The equivalent norm on X which we need, we define as follows: $\|x\|' = \lim_{\mathcal{U}} \|x\|_Y$. The expression under the limit is defined for $Y \in \mathbb{E}$ big enough, it is bounded, so the limit exists. It is evident, that $\|\cdot\|'$ is an equivalent norm on X, $\frac{1}{2}\|x\| \le \|x\|' \le 3\|x\|$. Let us prove, that $(X, \|\cdot\|')$ possesses the d-big slice property. Fix a slice S of the unit ball $B(X, \|\cdot\|')$, generated by a functional f and an $\varepsilon > 0$. Let $e \in S$,

$$f(e) > 1 - \frac{\varepsilon}{2}. \tag{12}$$

Denote by e_n, e_n^*, $n \in \mathbb{N}$ the canonical basic of c_0 and the corresponding coordinate functionals. Since $\{e_n\}$ tends weakly to 0, there is an $m \in \mathbb{N}$ with

$$|f(e_m)| < \frac{\varepsilon}{4}. \tag{13}$$

Consider elements of the form $e + te_m$. For every $Y \in \mathbb{E}$, $e \in Y$ we have

$$\|e+te_m\|_Y = \|P_Y(e+te_m)\| \bigvee \|e+te_m - P_Y(e+te_m)\| = \|P_Y(e)+te_m\| \bigvee \|e - P_Y e\|$$

$$= |t + e_m^*(P_Y e)| \bigvee \|P_Y e - e_m^*(P_Y e) e_m\| \bigvee \|e - P_Y e\|. \tag{14}$$

Denote $a = \lim_{\mathcal{U}} e_m^*(P_Y e)$, $q = \lim_{\mathcal{U}} \|P_Y e - e_m^*(P_Y e) e_m\| \bigvee \|e - P_Y e\|$ and remark, that a and q do not depend on t. According to (14)

$$\|e + t e_m\|' = \max\{|t + a|, q\}. \tag{15}$$

Since $e \in S$, we know that $|a| \le 1$ and $q \le 1$. By (15) the elements $x_1 = e + (1 - a) e_m$, $x_2 = e - (1 + a) e_m$ belong to the unit ball $B(X, \|\cdot\|')$, and due to (12) and (13), $f(x_j) \ge 1 - \varepsilon, j = 1, 2$. Hence $x_1, x_2 \in S$. To complete the proof it is sufficient to notice that $\|x_1 - x_2\|' = 2$. ∎

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