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Complete Solution of an Inverse Problem for One Class of the High Order Ordinary Differential Operators with Periodic Coefficients

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The purpose of the present work is to solve the characterization problem, which consists of identification of necessary and sufficient conditions on the scattering data ensuring that the reconstructed potential belongs to particular class.

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1. Introduction

The purpose of the present work is solving the characterization problem, which consists of identification of necessary and sufficient conditions on the scattering data ensuring that the reconstructed potential belongs to a particular class. In our case Q^2 is the class of all 2π-periodic complex-valued functions on the real axis R, belonging to the space $L_2[0, 2\pi]$, and Q_+^2 is its subclass consisting of the functions

$$p_\gamma(x) = \sum_{n=1}^{\infty} p_{\gamma n} exp\,(inx), \quad \sum_{\gamma=0}^{2m-2} \sum_{n=1}^{\infty} n^\gamma \left| p_{\gamma n} \right| < \infty. \tag{1.1}$$

The object under consideration is the operator L, generated by the differential expression

$$l(y) = (-1)^m y^{(2m)} + \sum_{\gamma=0}^{2m-2} p_\gamma(x) y^{(\gamma)}(x) \tag{1.2}$$

in the space $L_2(-\infty,\infty)$, with the coefficients $p_\gamma(x)\in Q_+^2$.

Note, that some of the characterizations for the Sturm–Liouville operators in the real-valued potentials belonging to the $L_1^1(R)$ ($L_\alpha^1(R)$ is the class of measurable potentials satisfying the condition $\int\limits_R dx(1+|x|)^\alpha |p_\gamma(x)|<\infty$), have been given by A. Melin [1] and V.A. Marchenko [2]. More details review can be found in the papers [3–5].

The inverse problem for the coefficients (1.1) for the first time was formulated and solved in paper [6], where it was shown that the equation $l(y)=\lambda^{2m}y$ has the solution

$$\varphi(x,\lambda\omega_\tau)=e^{i\lambda\omega_\tau x}+\sum_{j=1}^{2m-1}\sum_{\alpha=1}^{\infty}\sum_{n=1}^{\alpha}\frac{V_{n\alpha}^{(j)}}{n+\lambda\omega_\tau(1-\omega_j)}e^{(i\lambda\omega_\tau+i\alpha)x},\ \tau=\overline{0,2m-1},$$

$$\omega_j=exp\,(ij\pi/m)\,. \tag{1.3}$$

and Wronskian of the system of solutions $\varphi(x,\lambda\omega_\tau)$ being equal to $(i\lambda)^{m(2m-1)}A$, where

$$A=\begin{vmatrix}1 & 1 & .. & 1\\ \omega_1 & \omega_2 & .. & \omega_{2m-1}\\ .. & .. & .. & ..\\ \omega_1^{2m-1} & \omega_2^{2m-1} & .. & \omega_{2m-1}^{2m-1}\end{vmatrix}$$

is nonzero if $\lambda\neq 0$.

The limit $\varphi_{nj}(x)\equiv\lim\limits_{\lambda\to-\lambda_{nj}}(\lambda+\lambda_{nj})\,\varphi(x,\lambda)$, $\lambda_{nj}=-\frac{n}{1-\omega_j}$, $n\in N$, $j=\overline{1,2m-1}$, is also a solution of the equation $l(y)=\lambda^{2m}y$ but already linearly depending on $\varphi(x,\lambda_{nj}\omega_j)$. Therefore, there exist the numbers $\tilde S_{nj}, n\in N, j=\overline{1,2m-1}$, for which the conditions

$$\varphi_{nj}(x)=\tilde S_{nj}\varphi(x,\lambda_{nj}\omega_j) \tag{1.4}$$

are fulfilled.

It was established by M.G. Gasymov [6] that if

I. $\sum\limits_{n=1}^{\infty} n\left|\tilde S_n\right|<\infty$,

II. $4^{m-1}a_m\sum\limits_{n=1}^{\infty}\frac{|\tilde S_n|}{n+1}=p<1$, where

$$a_m=\max_{\substack{1\le j\le l\le 2m-1\\ 1\le n,r<\infty}}\frac{|(1-\omega_j)\ (n+r)|}{|\,r\,(1-\omega_j)\ -\ n\,(1-\omega_l)\ \omega_j\,|},\quad \tilde S_n=\sum_{j=1}^{2m-1}n^{2m-2}|\tilde S_{nj}|, \tag{1.5}$$

then there exist the uniquely defined functions $p_\gamma(x)$, $\gamma=\overline{0,2m-2}$ of (1.1), for which the numbers $\{\tilde S_n\}$ are defined by formulae (1.3)–(1.4). Then the complete solution of this problem at $m = 1$ was given in paper [3], where the authors proved the following theorem:

Theorem 1. *In order the given sequence of complex numbers* $\{\hat{S}_n\}$ *to be a set of spectral data of the operator* $L = \left(\frac{d}{dx}\right)^2 + p_0(x)$ *with the potential* $p_0(x) \in Q_+^2$ *it is necessary and sufficient, that the following conditions are fulfilled:*

1) $\{n\hat{S}_n\}_{n=1}^{\infty} \in l_2$;

2) the infinite determinant $D(z) \equiv \left\| \delta_{nk} + \frac{2\hat{S}_k}{n+k} e^{i\frac{n+k}{2}z} \right\|_{n,k=1}^{\infty}$ *exists* (δ_{nk} *is Kronecker's symbol), is continuous, not equal to zero in the closed half-plane* $\overline{C_+} = \{z : \operatorname{Im} z \geq 0\}$ *and analytical inside of the open half-plane* $C_+ = \{z : \operatorname{Im} z > 0\}$.

In the present work the complete inverse problem (characterization problem) for the high order ordinary differential operators (1.2) with the coefficients (1.1) is solved.

Let us formulate now the basic result of the present work.

Definition. *The sequence* $\{\tilde{S}_{nj}\}_{n=1,j=1}^{\infty,2m-1}$, *constructed by means of the formulae (1.4), is called a set of spectral data of the operator (1.2) with the coefficients (1.1).*

Theorem 2. *For a given sequence of complex numbers* $\{\tilde{S}_{nj}\}_{n=1,j=1}^{\infty,2m-1}$ *to be a set of spectral data of the operator L, generated by the differential expression (1.2) and coefficients (1.1), it is necessary and sufficient that the following conditions are fulfilled:*

$$\{n\tilde{S}_n\}_{n=1}^{\infty} \in l_1; \tag{1.6}$$

2) the infinite determinant

$$D(z) \equiv det \left\| \delta_{rn} E_{2m-1} - \left\| \frac{i(1-\omega_l)\tilde{S}_{nj}}{r\omega_l(1-\omega_j) - n(1-\omega_l)} e^{i\frac{n}{1-\omega_j}z} e^{-i\frac{r\omega_l}{1-\omega_l}z} \right\|_{j,l=1}^{2m-1} \right\|_{r,n=1}^{\infty} \tag{1.7}$$

exists, (E_n *is the unit* $n \times n$ *matrix), is continuous, not equal to zero in the close half-plane* $\overline{C_+} = \{z : \operatorname{Im} z \geq 0\}$, *and analytical inside of the open half-plane* $C_+ = \{z : \operatorname{Im} z > 0\}$.

2. On the inverse problem of scattering theory on the semiaxis

On the base of the proof of the Theorem 2 we will study the equation $l(y) = \lambda^{2m} y$.

Denoting

$$x = it, \lambda = -ik, y(x) = Y(t), \tag{2.1}$$

we obtain the equation

$$(-1)^m Y^{(2m)}(t) + \sum_{\gamma=0}^{2m-2} Q_\gamma(t)\, Y^{(\gamma)}(t) = k^{2m} Y(t), \tag{2.2}$$

in which

$$Q_\gamma(t) = (-1)^m (-i)^\gamma \sum_{n=1}^{\infty} p_{\gamma n} e^{-nt}, \quad \sum_{\gamma=0}^{2m-2} \sum_{n=1}^{\infty} n^\gamma |p_{\gamma n}| < \infty. \tag{2.3}$$

As a result we obtain the equation (2.1) whose coefficient exponentially decrease as $t \to \infty$.

Lemma 1. *The kernel of the transformation operator of equation (2.2) $K(t,u)$, $u \geq t$, attached to $+\infty$, with the coefficients (2.3) permits the representation*

$$K(t,u) = \sum_{j=1}^{2m-1} \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}^{(j)}}{i(1-\omega_j)} e^{-\alpha t + \frac{n}{1-\omega_j}(t-u)},$$

in which the series

$$\sum_{j=1}^{2m-1} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\alpha=n}^{\infty} \alpha^{2m-1} (\alpha - n) \left| V_{n\alpha}^{(j)} \right|,$$

$$\sum_{j=1}^{2m-1} \sum_{\alpha=1}^{\infty} \alpha^{2m-1} \left| V_{\alpha\alpha}^{(j)} \right|$$

are convergent.

P r o o f. It is shown in [7] that equation (2.2) with the coefficients (2.3) has the solution

$$f(t, k\omega_\tau) = e^{ik\omega_\tau t} + \sum_{j=1}^{2m-1} \sum_{\alpha=1}^{\infty} \sum_{n=1}^{\alpha} \frac{V_{n\alpha}^{(j)}}{in + k\omega_\tau (1-\omega_j)} e^{(ik\omega_\tau - \alpha)t}, \quad \tau = 0, 2m-1, \tag{2.4}$$

and the numbers $V_{n\alpha}^{(j)}$ are defined from the following recurrent formulae

$$\left[\left(\alpha - \frac{n}{(1-\omega_j)} \right)^{2m} - \left(\frac{n}{(1-\omega_j)} \right)^{2m} \right] V_{n\alpha}^{(j)}$$

$$= (-1)^{m+1} \sum_{\gamma=0}^{2m-1} \sum_{s=n}^{\alpha-1} \left[i \left(s - \frac{n}{(1-\omega_j)} \right) \right]^\gamma P_{\gamma, s-n} V_{ns}^{(j)} \tag{2.5}$$

at

$$\alpha = 2, 3, \ldots;\ n = 1, 2, \ldots, \alpha - 1;\ j = 1, 2, \ldots, 2m - 1,$$

$$i^{\gamma} p_{\gamma\alpha} + \sum_{j=1}^{2m-1} \sum_{n=1}^{\alpha} d_{j\gamma}(n, \alpha) V_{n\alpha}^{(j)} + \sum_{\nu=\gamma+1}^{2m-2} \sum_{j=1}^{2m-1} \sum_{r+s=\alpha} \sum_{n=1}^{s} d_{j\gamma}(n, s, \nu) p_{\nu r} V_{ns}^{(j)} = 0, \tag{2.6}$$

where

$$\frac{1}{n + k(1 - \omega_j)} \left[(i\alpha + k)^{2m} - k^{2m} - (i\alpha + k_{nj})^{2m} + k_{nj}^{2m} \right]$$

$$= \sum_{\gamma=0}^{2m-2} d_{j\gamma}(n, \alpha) k^{\gamma};\ \ j = \overline{1, 2m - 1},$$

$$\frac{(is + k)^{\nu} - (is + k_{nj})^{\nu}}{in + k(1 - \omega_j)} = \sum_{\gamma=0}^{v-1} d_{j\gamma}(n, s, v) k^{\gamma},$$

and the series (2.4) permits 2m times term by term differentiation. Then according to conditions (2.3), we have

$$f(t, k) = e^{ikt} + \int_{t}^{\infty} K(t, u) e^{iku} du, \tag{2.7}$$

where

$$K(t, u) = \sum_{j=1}^{2m-1} \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}^{(j)}}{i(1 - \omega_j)} e^{-\alpha t + \frac{n}{1-\omega_j}(t-u)}. \tag{2.8}$$

The lemma is proved.

Then it is possible to get equality [8]

$$f_{nj}(t) = S_{nj} f(t, k_{nj}\omega_j), \tag{2.9}$$

where

$$f_{nj}(t) = \lim_{k \to k_{nj}} [in + k(1 - \omega_j)] f(t, k),\ k_{nj} = -\frac{in}{1 - \omega_j},\ j = \overline{1, 2m - 1},\ n \in N.$$

Rewriting equality (2.9) in the form

$$\sum_{\alpha=n}^{\infty} V_{n\alpha}^{(j)} e^{-\alpha t} e^{\frac{n}{1-\omega_j} t}$$

$$= S_{nj} e^{\frac{n\omega_j}{1-\omega_j} t} + \sum_{l=1}^{2m-1} \sum_{r=1}^{\infty} \sum_{\alpha=r}^{\infty} \frac{i(1 - \omega_j) V_{nr}^{(l)} S_{nj}}{n\omega_j (1 - \omega_l) - r(1 - \omega_j)} e^{\left(-\alpha + \frac{n\omega_j}{1-\omega_j}\right) t} \tag{2.10}$$

and denoting by

$$\tilde{F}(t+u) = \sum_{j=1}^{2m-1} \sum_{n=1}^{\infty} \frac{S_{nj}}{i(1-\omega_j)} e^{\frac{n}{1-\omega_j}(t\omega_j - u)}, \ t \le u, \tag{2.11}$$

we obtain the Marchenko type equation

$$K(t,u) = \tilde{F}(t+u) + \int_t^{\infty} K(t,s)\tilde{F}(s+u)ds \tag{2.12}$$

from equation (2.10). So, it is proved the following

Lemma 2. *If the coefficients $Q_\gamma(t)$ of equation (2.2) have form (2.3), then at every $t \ge 0$ the kernel of the transformation operator (2.8) satisfies to the equation of the Marchenko type (2.12) in which the transition function $\tilde{F}(t)$ has form (2.11), and the numbers S_{nj} are defined by equality (2.9), from which it is obtained, that $S_{nj} = V_{nn}^{(j)}$.*

The coefficients $Q_\gamma(t)$ are reconstructed by the kernel of the transformation operator by means of the recurrent formulae (2.5)–(2.6). Hence, the basic equation (2.12) and form of the transition function (2.11) make natural the formulation of the inverse problem for reconstruction of coefficients of equation (2.1) by numbers S_{nj}. In this formulation, which employs the transformation operator, an important moment is a proof of unique solubility of the basic equation (2.12).

Lemma 3. *The homogeneous equation*

$$g(s) - \int_0^{\infty} \tilde{F}(u+s)\, g(u)\, du = 0 \tag{2.13}$$

corresponding to the coefficients $Q_\gamma(t) \in Q_+^2$ has only a trivial solution.

P r o o f. Let $g \in L_2(R^+)$ be a solution of equation (2.13) and f be a solution of the equation

$$f(s) + \int_0^{s} K(t,s)\, f(t)\, dt = g(s). \tag{2.14}$$

Substituting g into (2.14) and taking into account equation (2.12), we get

$$f(s) + \int_0^{s} K(t,s)f(t)dt + \int_0^{\infty} [f(u) + \int_0^{u} K(t,u)f(t)dt]\tilde{F}(u+s)du$$

$$= f(s) + \int_s^\infty f(t)[\tilde{F}(t+s) + \int_t^\infty K(t,u)\tilde{F}(u+s)du] = 0.$$

As at $t \geq s$ the estimation

$$\left| \tilde{F}(t+s) + \int_0^\infty K(t,u)\,\tilde{F}(u+s)\,du \right| \leq Ce^{-s}$$

is fulfilled, hence it follows that $f = 0, g = 0$ and lemma is proved.

Lemma 4. *At every fixed value* $a, (\operatorname{Im} a \geq 0)$ *the homogeneous equation*

$$g(s) - \int_0^\infty \tilde{F}(u+s-2ia)g(u)du = 0 \tag{2.15}$$

has only a trivial solution in the space $L_2(R^+)$.

P r o o f. We substitute x by $x+a$, where $\operatorname{Im} a \geq 0$ in equation (1.2). Then we obtain the same equation with the coefficient $Q^a_\gamma(x) = Q_\gamma(x+a)$ satisfying condition (1.1). Let us remark, that the functions $\varphi(x+a, \lambda\omega_j)$ are solutions of the equation

$$(-1)^m y^{(2m)}(x) + \sum_{\gamma=0}^{2m-2} Q^a_\gamma(x)\, y^{(\gamma)}(x) = \lambda^{2m} y(x)$$

that at $x \to \infty$ have the form

$$\varphi(x+a, \lambda\omega_j) = e^{i\lambda\omega_j a} e^{i\lambda\omega_j x} + o(1).$$

Therefore the functions

$$\varphi^a(x, \lambda\omega_j) = e^{-i\lambda\omega_j a}\varphi(x+a, \lambda\omega_j)$$

are also solutions of type (1.3). Then let us denote by $S_{nj}(a)$ the spectral data of the operator L with the potential $Q^a_\gamma(x)$

$$L \equiv (-1)^m \frac{d^{2m}}{dx^{2m}} + \sum_{\gamma=0}^{2m-2} Q^a_\gamma(x) \frac{d^\gamma}{dx^\gamma}.$$

According to (1.4), we have

$$S_{nj}(a)\,\varphi^a(x, \lambda_{nj}\omega_j) = \lim_{\lambda \to \lambda_{nj}} [n + \lambda(1-\omega_j)]\,\varphi^a(x, \lambda)$$

$$= \lim_{\lambda \to \lambda_{nj}} [n + \lambda(1-\omega_j)] \varphi(x+a,\lambda)\, e^{-i\lambda a}$$

$$= e^{i\frac{n}{1-\omega_j}a} S_{nj} \varphi\left(x+a, \lambda_{nj}\omega_j\right) = e^{i\frac{n}{1-\omega_j}a}\, e^{-i\frac{n\omega_j}{1-\omega_j}a} S_{nj} \varphi^a\left(x, \lambda_{nj}\omega_j\right)$$

$$= e^{ina} S_{nj} \varphi^a\left(x, \lambda_{nj}\omega_j\right).$$

Hence

$$S_{nj}(a) = e^{ina} S_{nj}. \tag{2.16}$$

Now arguing as above, we obtain the basic equation of form (2.12) with the transition function

$$\tilde{F}_a(t+u) = \sum_{j=1}^{2m-1} \sum_{n=1}^{\infty} \frac{S_{nj}(a)}{i(1-\omega_j)} e^{\frac{n}{1-\omega_j}(t\omega_j - u)} = \tilde{F}(t - ia + u - ia) = \tilde{F}(t + u - 2ia).$$

From this lemma follows

Theorem 3. *The coefficients $Q_\gamma(t)$ of equation (2.1), satisfying to condition (2.2), are uniquely defined by the numbers S_{nj}.*

3. Proof of Theorem 2

Necessity. From the relation (2.9) and form of the function $f_{nj}(t)$ we obtained that

$$S_{nj} = V_{nn}^{(j)}.$$

Therefore

$$\sum_{j=1}^{2m-1} \sum_{n=1}^{\infty} n^{2m-1} \left|S_{nj}\right| \le \sum_{j=1}^{2m-1} \sum_{n=1}^{\infty} n^{2m-1} \left|V_{nn}^{(j)}\right| < \infty,$$

i.e., $n^{2m-1}\left|S_{nj}\right| \in l_1$. The necessity of condition (1) is proved.

To proof the necessity of condition (2) let us demonstrate first of all that from the trivial solubility of the basic equation (2.12) at $t = 0$ in the class of functions, satisfying to the inequality $\|g(u)\| \le Ce^{-\frac{u}{2}}$, $u \ge 0$, follows the trivial solubility in $l_2\left(1, \infty, R^{2m-1}\right)$ of the infinite system of equations

$$g_{jn} - \sum_{l=1}^{2m-1} \sum_{r=1}^{\infty} \frac{i(1-\omega_j) S_{rl}}{n\omega_j(1-\omega_l) - r(1-\omega_j)} g_{lr} = 0. \tag{3.1}$$

Really, if $\{g_{jn}\} \in l_2$, $j = \overline{1, 2m-1}$, is a solution of this system, then the function

$$g(u) = \left(g_1(u), g_2(u), \ldots, g_{2m-1}(u)\right) = \sum_{j=1}^{2m-1} \sum_{n=1}^{\infty} S_{nj} g_{jn} e^{-\frac{n}{1-\omega_j}u}, \tag{3.2}$$

defined for all $u \geq 0$, satisfies to the inequality

$$|g(u)| \leq Ce^{-\frac{u}{1-\omega_j}};\ u \geq 0,$$

and is a solution of equation (2.13)

$$g(u) - \int_0^\infty g(s)\tilde{F}(u+s)ds = \sum_{j=1}^{2m-1}\sum_{n=1}^{\infty} S_{nj}g_{jn}e^{-\frac{n}{1-\omega_j}u}$$

$$-\int_0^\infty (\sum_{j=1}^{2m-1}\sum_{n=1}^{\infty} S_{nj}g_{jn}e^{-\frac{n}{1-\omega_j}s})(\sum_{l=1}^{2m-1}\sum_{r=1}^{\infty}\frac{S_{rl}}{i(1-\omega_j)}e^{\frac{r}{1-\omega_l}(s\omega_l-u)})ds$$

$$= \sum_{j=1}^{2m-1}\sum_{n=1}^{\infty} S_{nj}g_{jn}e^{-\frac{n}{1-\omega_j}u}$$

$$-\sum_{j=1}^{2m-1}\sum_{n=1}^{\infty}\sum_{l=1}^{2m-1}\sum_{r=1}^{\infty}\frac{S_{nj}S_{rl}}{i(1-\omega_l)}g_{jn}e^{-\frac{r}{1-\omega_l}u}\int_0^\infty e^{-\frac{n}{1-\omega_j}s}e^{\frac{r\omega_l}{1-\omega_l}s}ds$$

$$= \sum_{j=1}^{2m-1}\sum_{n=1}^{\infty} S_{nj}g_{jn}e^{-\frac{n}{1-\omega_j}u}$$

$$-\sum_{j=1}^{2m-1}\sum_{n=1}^{\infty}\sum_{l=1}^{2m-1}\sum_{r=1}^{\infty}\frac{i(1-\omega_j)S_{nj}S_{rl}}{n\omega_j(1-\omega_l)-r(1-\omega_j)}g_{lr}e^{-\frac{n}{1-\omega_j}u}$$

$$= \sum_{j=1}^{2m-1}\sum_{n=1}^{\infty} S_{nj}e^{-\frac{n}{1-\omega_j}u}[g_{jn} - \sum_{l=1}^{2m-1}\sum_{r=1}^{\infty}\frac{i(1-\omega_j)S_{rl}}{n\omega_j(1-\omega_l)-r(1-\omega_j)}g_{lr}] = 0.$$

Since, $g(u) = 0$, then $S_{nj}g_{jn} = 0$ for all $n \geq 1$, $j = \overline{1, 2m-1}$, and $g_{jn} = 0$, $j = \overline{1, 2m-1}$, $n \geq 1$, according to (3.1). Let us introduce in the space $l_2\left(1, \infty; R^{2m-1}\right)$ the operator $F(t)$, given by the matrix

$$F_{rn}(t) = \left\|F_{rn}^{jl}\right\|_{j,l=1}^{2m-1} = \left\|\frac{i(1-\omega_l)S_{nj}}{r\omega_l(1-\omega_j) - n(1-\omega_l)}e^{-\frac{n}{1-\omega_j}t}e^{\frac{r\omega_l}{1-\omega_l}t}\right\|_{j,l=1}^{2m-1}, \quad (3.3)$$

and let $\varphi_{2k-1} = \{(\delta_{\nu 1})\,\delta_{kr}\}_{\nu,r=1}^{2m-1,\infty}$, $\varphi_{2k} = \{(\delta_{\nu 2})\,\delta_{kr}\}_{\nu,r=1}^{2m-1,\infty}$ ((δ_{ij}) is a column vector) be the orthonormal system in this space. Then we obtain from $n^{2m-1}|S_{nj}| \in l_1$ $\sum_{j,k=1}^{\infty}\left|(F\varphi_j, \varphi_k)_{l_2(1,\infty;R^{2m-1})}\right| < \infty$, i.e., $F(t)$ is a kernel operator [9]. Therefore, there exists the determinant $\Delta(t) = det\,(E - F(t))$ of the operator $E - F(t)$, connected, as easy to see, with the determinant $D(z)$ from condition 2) of the Theorem 2, with relation $\Delta(-iz) = det(E - F(-iz)) \equiv D(z)$.

The determinant of system (3.1) is $D(0)$ and determinant of the similar system corresponding to the coefficient $Q_\gamma^z = Q_\gamma(x+z)$, $\operatorname{Im} z \geq 0$, is

$$D(z) \equiv \det \left\| \delta_{rn} E_{2m-1} - \left\| \frac{i(1-\omega_l)S_{nj}(z)}{r\omega_l(1-\omega_j) - n(1-\omega_l)} \right\|_{j,l=1}^{2m-1} \right\|_{r,n=1}^{\infty}$$

$$= \det \left\| \delta_{rn} E_{2m-1} - \left\| \frac{i(1-\omega_l)S_{nj}}{r\omega_l(1-\omega_j) - n(1-\omega_l)} e^{inz} \right\|_{j,l=1}^{2m-1} \right\|_{r,n=1}^{\infty}$$

$$= \det \left\| \delta_{rn} E_{2m-1} - \left\| \frac{i(1-\omega_l)S_{nj}}{r\omega_l(1-\omega_j) - n(1-\omega_l)} e^{i\frac{n}{1-\omega_j}z} e^{-i\frac{r\omega_l}{1-\omega_l}z} \right\|_{j,l=1}^{2m-1} \right\|_{r,n=1}^{\infty}.$$

Therefore, in order to prove the necessity of condition 2) of Theorem 2 one should check that $\Delta(0) = D(0) \neq 0$. System (3.1) can be written in $l_2\left(1,\infty; R^{2m-1}\right)$ as the equation

$$g - F(0)\, g = 0.$$

As $F(0)$ is the kernel operator, we can apply to this equation the Fredholm theory, according to which its trivial solvability is equivalent to condition that $det(E - F(0))$ is not equal to zero [10]. The necessity of condition 2) is proved.

Sufficiency. Let us multiply equation (2.12) by $e^{\frac{r\omega_l}{1-\omega_l}u}$ and integrate it over $u \in [t,\infty)$. We obtain

$$k(t) = F(t)\, e(t) + k(t)\, F(t), \tag{3.4}$$

in which the operator $F(t)$ is defined by the matrix $\|F_{rn}(t)\|_{r,n=1}^{\infty}$of the form (3.3),

$$e(t) = \|e_{nj}(t)\|_{n,j=1}^{\infty,2m-1} = \left\| e^{\frac{n\omega_j}{1-\omega_j}t} \right\|_{n,j=1}^{\infty,2m-1};$$

$$k(t) = \|k_{lr}(t)\|_{l,r=1}^{2m-1,\infty} = \left\| \int_t^\infty K(t,u)\, e^{\frac{r\omega_l}{1-\omega_l}u} du \right\|_{l,r=1}^{2m-1,\infty}.$$

As $F(t)$ is the trace class for $t \geq 0$ and the condition $\Delta(t) = det(E - F(t)) \neq 0$ holds, there exists the bounded in l_2 inverse operator $R(t) = (E - F(t))^{-1}$. Since $F(t)\, e(t) \in l_2$, then from (3.4) we get

$$k(t) = R(t)F(t)e(t). \tag{3.5}$$

Now denoting $\langle f,g\rangle = \sum\limits_{n=1}^{\infty} f_n g_n$, we find from (2.12) that

$$K(t,u) = \langle e(t), A(u)\rangle + \langle k(t), A(u)\rangle = \langle e(t), A(u)\rangle + \langle R(t)F(t)e(t), A(u)\rangle$$

$$= \langle e(t) + R(t)F(t)e(t), A(u)\rangle = \langle R(t)e(t), A(u)\rangle, \tag{3.6}$$

where $A(u)$ is defined by the matrix

$$A(u) = \|a_{jn}(u)\|_{j,n=1}^{2m-1,\infty} = \left\| \frac{S_{nj}}{i(1-\omega_j)} e^{-\frac{n}{1-\omega_j}u} \right\|_{j,n=1}^{2m-1,\infty}.$$

Now assume that the conditions of the theorem are fulfilled. According to the stated considerations, define the function $K(t,u)$ at $0 \le t \le u$ by equality (3.6). Then at $u \ge t$ we have

$$K(t,u) - \int\limits_t^{\infty} K(t,s)F(s+u)\,ds$$

$$= \langle R(t)e(t), A(u)\rangle - \int\limits_t^{\infty} \langle R(t)e(t), A(s)\langle e(s), A(u)\rangle\rangle ds$$

$$= \langle R(t)e(t), A(u)\rangle - \left\langle R(t)e(t), \left\langle \int\limits_t^{\infty} A(s)e(s)ds, A(u)\right\rangle\right\rangle = \langle R(t)e(t), A(u)\rangle$$

$$-\langle R(t)e(t), \langle A(u), F(t)\rangle\rangle = \langle R(t)e(t), A(u)\rangle - \langle R(t)e(t), A(u)F^*(t)\rangle$$

$$= \langle R(t)e(t), A(u) - A(u)F^*(t)\rangle = \langle e(t), A(u)\rangle = F(t+u),$$

where the symbol „$*$ " means transition to the matrix, adjoint to the $F(t)$ with respect to the bilinear form $\langle .\,,\ .\rangle$. So, we have

Lemma 5. *For any $t \ge 0$ the kernel $K(t,u)$ of the transformation operator satisfies to the basic equation*

$$K(t,u) = \tilde{F}(t+u) + \int\limits_t^{\infty} K(t,s)\tilde{F}(s+u)\,ds.$$

From Lemma 3 it follows unique solubility of the basic equation. By the direct substitution it is easy to calculate that the solution of the basic equation is

$$K(t,u) = \sum_{j=1}^{2m-1}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}^{(j)}}{i(1-\omega_j)} e^{-\alpha t + \frac{n}{1-\omega_j}(t-u)},$$

where the numbers $V_{n\alpha}^{(j)}$ are defined from the recurrent relations

$$V_{nn}^{(j)} = S_{nj}\,,$$

$$V_{n\alpha+n}^{(j)} = (1-\omega_j)V_{nn}^{(j)} \sum_{l=1}^{2m-1} \sum_{r=1}^{\alpha} \frac{V_{r\alpha}^{(l)}}{r(1-\omega_j) - n(1-\omega_l)\omega_j}\,.$$

Passing to the proof of the basic statement that the coefficients $Q_\gamma(t)$ have form (2.2), let us first establish the estimations for the matrix elements $R_{rn}(t)$ of the operator $R(t)$

$$\left|R_{rn}^{jl}(t)\right| \le \delta_{rn}\delta_{jl} + CS_n\,, \tag{3.7}$$

$$\left|\frac{\partial^{2m-\tau}}{\partial t^{2m-\tau}} R_{rn}^{jl}(t)\right| \le CS_n; \tau = \overline{1, 2m-1}, \tag{3.8}$$

where $C = max\{C_k > 0, k = \overline{1, 2m-1}\}$ is a constant, and $S_n = \sum\limits_{j=1}^{2m-1} n^{2m-1} |S_{nj}|$. Indeed, it follows from the identity $R(t) = E + R(t)F(t)$ that

$$\left|R_{rn}^{lj}(t)\right| \le \delta_{rn}\delta_{lj} + \sum_{\tau=1}^{2m-1}\sum_{p=1}^{\infty} \left|R_{rp}^{l\tau}(t)\right| \left|F_{pn}^{\tau j}(t)\right|$$

$$\le \delta_{rn}\delta_{lj} + 2\sum_{\tau=1}^{2m-1} (\sum_{p=1}^{\infty} \left|R_{rp}^{l\tau}(t)\right|^2)^{\frac{1}{2}} (\sum_{p=1}^{\infty} \left|F_{pn}^{\tau j}(t)\right|^2)^{\frac{1}{2}}$$

$$\le \delta_{rn}\delta_{lj} + C_1 a_m ((R(t)R^*(t))_{pp} \sum_{p=1}^{\infty} \frac{1}{(n+p)^2})^{\frac{1}{2}} S_n \le \delta_{rn}\delta_{lj} + C_1 \|R(t)\|_{l_2 \to l_2} S_n.$$

On the other hand, as it has been noted above, the operator-function $R(t) = (E - F(t))^{-1}$ exists and is bounded in the l_2(because $F(t)$ is the kernel operator at $t \ge 0$ and $\Delta(t) = det(E - F(t)) \ne 0$) that proves first inequality (3.7).

In order to proof the second estimation (3.8) we first obtain

$$\left|\frac{d}{dt} R_{rn}^{lj}(t)\right| \le \sum_{\tau=1}^{2m-1} \sum_{p,q=1}^{\infty} \left|R_{rp}^{l\tau}(t)\right| \left|\frac{d}{dt} F_{pq}^{\tau k}(t)\right| \left|R_{qn}^{kj}(t)\right|$$

$$\le \sum_{\tau=1}^{2m-1} \sum_{p,q=1}^{\infty} (\delta_{rp}\delta_{l\tau} + C_2 S_p) S_q \left(\delta_{qn}\delta_{kj} + C_3 S_n\right)$$

$$\le (1 + C_4 \sum_{p=1}^{\infty} S_p)^2 S_n \le CS_n.$$

Then from the equality $\frac{d^l}{dt^l}R(t) = \sum\limits_{n=1}^{l} C_l^n \left(\frac{d^{l-n}}{dt^{l-n}}R(t)\right)\left(\frac{d^n}{dt^n}F(t)\right)R(t)$, $l=\overline{1,2m-1}$, by the help of mathematical induction the inequality (3.8) is proved. In [11] the following relations were proved (for correspondence to our case, assume that $q_{2m-2-\gamma}(x) = Q_\gamma(x)$):

$$(-1)^m \frac{\partial^{2m}}{\partial x^{2m}}K(x,t) + \sum_{\gamma=0}^{2m-2} q_{2m-2-\gamma}(x)\frac{\partial^\gamma}{\partial x^\gamma}K(x,t) - \frac{\partial^{2m}}{\partial t^{2m}}K(x,t) = 0,$$

$$q_0(x) = 2m\frac{d}{dx}K(x,x),$$

$$q_{k+1}(x) = \sum_{\nu=0}^{k} q_\nu(x)\sum_{s=\nu}^{k} C_{2m-3-s}^{k-s}\left\{\frac{\partial^{s-\nu}}{\partial x^{s-\nu}}K(x,t)\Big|_{t=x}\right\}^{(k-s)}$$

$$+\sum_{k=0}^{k+2} C_{2m-1-\nu}^{k+2-\nu}\left\{\frac{\partial^\nu}{\partial x^\nu}K(x,t)\Big|_{t=x}\right\}^{(k+2-\nu)} - (-1)^k\frac{\partial^{k+2}}{\partial t^{k+2}}K(x,t)\Big|_{t=x},$$

$$k = 0,1,\ldots,2m-3.$$

Now it is easy to show, that

$$q_0(x) = \sum_{j=1}^{2m-1}\sum_{n=1}^{\infty}\frac{n\cdot S_{nj}}{i(1-\omega_j)}e^{-nt} + \Pi_0(t),$$

where $\Pi_0(t) = \sum\limits_{n,p,q,\tau=1}^{\infty} R_{np}^{ej}F_{pq}^{j\tau}e_{q\tau}e_{\tau q} = < R(t)F(t)e(t), A(t) >$ is $2i\pi$ periodic function and has bounded derivative until (2m-1) order. It follows from this fact that the Fourier coefficients of the function $\Pi_0(-ix)$, $x\in R$, are such that $\sum\limits_{n=1}^{\infty}\left|n^{2m-1}\Pi_n\right|^2<\infty$. But then $\sum\limits_{n=1}^{\infty} n^{2m-2}|\Pi_n|<\infty$. So, the Fourier coefficients $P_{2m-2,n}$ of the function $Q_{2m-2}(x) = q_0(x)$ satisfy condition (2.2). Similarly, for all other coefficients $Q_\gamma(x)$, $\gamma=\overline{0,2m-3}$, it is established that the Fourier coefficients $p_{\gamma n}$ of the function $Q_\gamma(x) = q_{2m-2-\gamma}(x)$, $\gamma=\overline{0,2m-3}$, satisfy condition (2.2). And it means that the Fourier coefficients of the function $P_\gamma(x)$, $\gamma=\overline{0,2m-2}$ satisfy condition (1.1).

Let, finally, $\left\{\tilde{S}_{nj}\right\}$ be the spectral data set of the operator $\left(L-k^{2m}E\right)$ with the constructed coefficients $P_\gamma(x)$. For completing of the proof it remains to show that $\{S_{nj}\}$ coincide with the initial set $\left\{\tilde{S}_{nj}\right\}$. This follows from the equality $\tilde{S}_{nj} = V_{nn}^{(j)} = S_{nj}$. The theorem is proved.

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