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On Existence of a Regular Hypersimplex Inscribed into the $(4n-1)$-Dimensional Cube

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It is proved the existence of a regular hypersimplex inscribed into the $(4n-1)$-dimensional cube under condition that some system of $4n-2$ algebraic equations with $4n-2$ unknown quantities $y_0, y_0', y_1, y_1', \ldots, y_{2n-2}, y_{2n-2}'$ has at least one solution with a real value of y_0 or any other $y_i \neq 0$, $i \geq 1$.

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1. Introduction

We obtained in [1] necessary and sufficient conditions for the existence of Hadamard's matrix of order $4n$ of the half-circulant type, which had introduced in [2], and announced the existence theorem for a regular hypersimplex inscribed into the $(4n-1)$-dimensional cube. The theorem will be proved in this paper. But first we recall some results from [1] and their consequences.

Convex n-gons P and P', inscribed into the regular $(2n-1)$-gon, are said to be *antipodal*, if the total number of their diagonals and sides of the same length equals n for all, without exception, admissible lengths. If the long radius of the regular $(2n-1)$-gon is 1 and its centre coincides with the origin of coordinates of the complex plane, it is no loss of generality to assume that its vertices are monomial z^k, $k = 0, 1, 2, \ldots, 2n-2$, where $z = e^{\frac{2\pi i}{2n-1}}$. Then n-gon P is represented by the *generating polynomial* $p(z) = \sum_{k=0}^{2n-2} x_k z^k$, where $x_k = 1$ if the vertex of the regular $(2n-1)$-gon with number k belongs to P, and $x_k = 0$ in otherwise.

Accordigly, n-gon P' is represented by a polynomial $p'(z) = \sum_{k=0}^{2n-2} x'_k z^k$. It follows from Lemma 1 in [1] that

$$|p|^2 = n + 2\sum_{k=1}^{n-1} d_k \cos\frac{2\pi k}{2n-1},$$

where d_k is the number of equal diagonals and sides of P, for which the vision angle (from the regular $(2n-1)$-gon centre) equals $\varphi_k = \frac{2\pi k}{2n-1}$, $k = 1, 2, \dots, n-1$. There is similar equality (with replacement d_k by d'_k) for the generating polynomial $p'(z)$. Since for antipodal n-gons P and P' by definition $d_k + d'_k = n$, $1 \le k \le n-1$, their generating polynomials satisfy relation $|p|^2 + |p'|^2 = n$ by Theorem 3 from [1].

Hadamard's matrix H of order $4n$ (every its entry equals 1 or -1 and its rows are pairwise orthogonal) is said to be *half-circulant* if it has the following form:

$$H = \begin{pmatrix} 1 & \cdots & 1 & \cdots \\ \vdots & A & \vdots & B \\ 1 & \cdots & -1 & \cdots \\ \vdots & B & \vdots & -A \end{pmatrix}, \tag{1}$$

where A and B are square matrices of order $2n-1$, being by circulants [2]. More precisely, A is an usual circulant [3, p. 272], which we will call the right circulant. And B is the left circulant, obtained (in contrast to A) by the cyclic permutation of entries of each subsequent row, starting from the first, on the left (but not to the right as in usual circulant). Hence, all the entries of the principal diagonal of the right circulant equal the first entry of the first row, while all the entries of the secondary diagonal of the left circulant equal to the last entry of the same row.

In [1, p. 58] the following necessary and sufficient condition for the existence of half-circulant Hadamard matrix of order $4n$ was obtained.

Theorem 1. *The half-circulant Hadamard matrix of order $4n$ exsists if and only if there exist antipodal n-gons P and P' inscribed into the regular $(2n-1)$-gon.*

Recall that the right circulant $A = (a_{im})$, $1 \le i, m \le 2n-1$ is determined by coefficients x_k, $k = 0, 1, 2, \dots, 2n-2$ of the polynomial $p(z)$, which represents P, while the left circulant $B = (b_{im})$, $1 \le i, m \le 2n-1$ is determined by coefficients x'_k of the $p'(z)$, which represents P', namely:

$$a_{im} = 1 - 2x_{|m-i|}, \qquad b_{im} = 1 - 2x'_{|m+i-2|}. \tag{2}$$

Here the sign of modulus means the least nonnegative residue modulo $2n-1$.

The rows of the matrix $\bar{H}$, obtained from Hadamard matrix H by removing its first column (consisting from $+1$ only), are, obviously, coordinates of vertices of a regular hypersimplex in E^{4n-1}, inscribed into a hypercube with edge 2, whose centre coincides with the origin. This implies

Corollary. *In order that one can inscribed a regular hypersimplex into a cube of dimension* $4n-1$ *it is sufficient that there exist antipodal convex* n*-gons inscribed in the regular* $(2n-1)$*-gon.*

Theorem 1 gives geometrical necessary and sufficient conditions for the existence of half-circulant Hadamard matrices. To find adequate analitical conditions a homogeneous polynomial of third degree $w = w(y) = w_0^3 + \sum_{m=1}^{n-1}(w_m^3 + w_{2n-1-m}^3)$ was introduced in [1]. Here y is the vector with coordinates y_0, $y_1, \ldots$, y_{2n-2} and

$$\begin{aligned} w_0 &= \frac{1}{\sqrt{2n-1}}(y_0 + \sqrt{2}\sum_{j=1}^{n-1} y_j), \\ w_m &= \frac{1}{\sqrt{2n-1}}[y_0 + \sqrt{2}\sum_{j=1}^{n-1}(y_j \cos\frac{2\pi mj}{2n-1} + y_{2n-1-j}\sin\frac{2\pi mj}{2n-1})], \qquad (3) \\ w_{2n-1-m} &= \frac{1}{\sqrt{2n-1}}[y_0 + \sqrt{2}\sum_{j=1}^{n-1}(y_j \cos\frac{2\pi mj}{2n-1} - y_{2n-1-j}\sin\frac{2\pi mj}{2n-1})], \end{aligned}$$

and also the matrix of system (3) is orthogonal $T = (t_{ik})$, $det\, T = 1$. Thus, our system can be written in the form: $\quad w_i = \sum_{k=0}^{2n-2} t_{ik} y_k, \quad i = 0, 1, 2, \ldots, 2n-2.$

Note that in [1, Th. 6] it was established what the surface of third order $w = n$ is *a non-reducible smooth hypersurface* in the projective space P^{2n-1}. Another characteristic property of the polynomial $w(y)$ is as follows: the system of equations

$$y = \frac{1}{3}\nabla w, \qquad (4)$$

where ∇w is the vector with coordinates $\frac{\partial w}{\partial y_i}$, $i = 0, 1, 2, \ldots, 2n-2$, is *idempotent*. This means that the solution of (4) with respect to $w_i^2 = \frac{1}{3}\frac{\partial w}{\partial w_i}$ has the form: $w_i^2 = \sum_{k=0}^{2n-2} t_{ik} y_k = w_i$, i.e., every w_i is equal to its square, and consequently it can assume only values 0 or 1. This follows from the next property: the matrix of the arising linear system coincides with the trasposed matrix T' of system (3), which is inverse of T, as $det\, T = 1$ (see [1, p. 62]). Moreover, if in the left hand side of (4) the vector y with coordinate $y_0, y_1, \ldots, y_{2n-2}$ is replaced by a vector with coordinates $y_0 + \delta, y_1, \ldots, y_{2n-2}$, then *it follows from the idempotent property that the solution of system (4) with respect to* w_i^2 *has next form:* $w_i^2 = \sum_{k=0}^{2n-2} t_{ik} y_k + t_{i0}\delta = w_i + \frac{\delta}{\sqrt{2n-1}}$ *for all* $i = 0, 1, 2, \ldots, 2n-2$, because in (3) all coefficients t_{i0} at y_0 are equal to $\frac{1}{\sqrt{2n-1}}$. In particular, if $y_i = 0$ for all i, then $w_i^2 = \frac{\delta}{\sqrt{2n-1}}$ for $i = 0, 1, 2, \ldots, 2n-2$.

Assume that in (3) $w_i = x_i$, where x_i is 0 or 1, i.e., the values which every w_i can take for any solution $y_0, y_1, \ldots, y_{2n-2}$ of system (4). Then we may obtain the following solution of (3), viewed as a linear system with respect to unknown quantities $y_0, y_1, \ldots, y_{2n-2}$:

$$\begin{aligned} y_0 &= \frac{1}{\sqrt{2n-1}} \sum_{i=0}^{2n-2} x_i, \\ y_j &= \sqrt{\frac{2}{2n-1}} [x_0 + \sum_{m=1}^{n-1} (x_m + x_{2n-1-m}) \cos \frac{2\pi jm}{2n-1}], \\ y_{2n-1-j} &= \sqrt{\frac{2}{2n-1}} \sum_{m=1}^{n-1} (x_m - x_{2n-1-m}) \sin \frac{2\pi jm}{2n-1}. \end{aligned} \tag{5}$$

Since $x_i = 0$ or 1 for all $i = 0, 1, 2, \ldots, 2n-2$, it follows that system (4) has 2^{2n-1} solutions. Besides, all its solutions are real (see Lem. 2 in [1]). Among them there are C_{2n-1}^n solutions such that $\sum_{i=0}^{2n-2} x_i = n$. Obviously, these solutions induce convex n-gons, inscribed into the regular $(2n-1)$-gon, with generating polynomials $\sum_{k=0}^{2n-2} x_k z^k$.

Now, let $y = \{y_0, y_1, \ldots, y_{2n-2}\}$ and $\bar{y} = \{\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_{2n-2}\}$ be two solutions of system (4), moreover, the first one has form (5) and the second one is given by (5) in which x_i, $i = 0, 1, 2, \ldots, 2n-2$ are replaced by $\bar{x}_i$, where $\bar{x}_i = 1 - x_i$ (this is possible, since it was proved earlier that every x_i is 0 or 1, as well as $1 - x_i$).

Lemma 1. *If $y = \{y_0, y_1, \ldots, y_{2n-2}\}$ is a solution of idempotent system (4), then $\bar{y} = \{\sqrt{2n-1} - y_0, -y_1, \ldots, -y_{2n-2}\}$ is also its solution.*

Indeed, summing termwise equations (5) for solution y and similar equations for solution $\bar{y}$, we have in view of the conditions $x_i + \bar{x}_i = 1$ for all admissible i and the identity $\frac{1}{2} + \sum_{m=1}^{n-1} \cos \frac{2\pi jm}{2n-1} \equiv 0$ (see [1, p. 55]):

$$\begin{aligned} y_0 + \bar{y}_0 &= \frac{1}{\sqrt{2n-1}} \sum_{i=0}^{2n-2} (x_i + \bar{x}_i) = \sqrt{2n-1}, \\ y_j + \bar{y}_j &= \sqrt{\frac{2}{2n-1}} (1 + 2 \sum_{m=1}^{n-1} \cos \frac{2\pi jm}{2n-1}) = 0, \\ y_{2n-1-j} + \bar{y}_{2n-1-j} &= \sqrt{\frac{2}{2n-1}} \sum_{m=1}^{n-1} 0 \cdot \sin \frac{2\pi jm}{2n-1} = 0, \end{aligned}$$

whence the Lemma 1 follows.

Solutions of system (4), corresponding by (5) to sets $\{x_i\}$ and $\{\bar{x}_i\}$, for which $x_i + \bar{x}_i = 1$ at any admissible i, are said to be *conjugate*. They possess some property, which follows immediately from the relations obtained above.

Lemma 2. *If a solution* $y = \{y_0, y_1, \dots, y_{2n-2}\}$ *of idempotent system (4) satisfies the equality* $y_0 + \sum_{i=1}^{2n-2} \alpha_i y_i = 0$, *where each* α_i *is any real number, then the conjugate solution* $\bar{y} = \{\bar{y}_0, \bar{y}_1, \dots, \bar{y}_{2n-2}\}$ *satisfies the equality* $\bar{y}_0 + \sum_{i=1}^{2n-2} \alpha_i \bar{y}_i = \sqrt{2n-1}$. *Conversely, if* $y_0 + \sum_{i=1}^{2n-2} \alpha_i y_i = \sqrt{2n-1}$, *then* $\bar{y}_0 + \sum_{i=1}^{2n-2} \alpha_i \bar{y}_i = 0$.

It should be observed, that in [1, equalities (14)] formulas for all derivatives involved in (4), were given. We need now one of them, with index $i = 0$, namely:

$$\frac{\partial w}{\partial y_0} = \frac{3}{\sqrt{2n-1}} \sum_{i=0}^{2n-2} y_i^2, \tag{6}$$

which one can also obtain with the aid of (3).

Recall at last that by Lemma 3 from [1], convex n-gons P and P', inscribed in the regular $(2n-1)$-gon and represented by polynomials $p(z) = \sum_{k=0}^{2n-2} x_k z^k$ and $p'(z) = \sum_{k=0}^{2n-2} x'_k z^k$, are antipodal ones, if and only if the coordinates of the corresponding vectors y and y', represented by the equality (5) and similar equality with primes, satisfy for all $j = 1, 2, \dots, n-1$ *antipodal* conditions

$$y_j^2 + y_{2n-1-j}^2 + y_j'^2 + y_{2n-1-j}'^2 = \frac{2n}{2n-1}. \tag{7}$$

Hence we have obtained analytical necessary and sufficient conditions for the existence of Hadamard's matrices (see [1, Th. 5]).

Theorem 2. *A half-circulant Hadamard matrix of order* $4n$ *exists if and only if system (4) has two solutions* $y = \{y_0, y_1, \dots, y_{2n-2}\}$ *and* $y' = \{y'_0, y'_1, \dots, y'_{2n-2}\}$ *such that* $y_0 = y'_0 = \frac{n}{\sqrt{2n-1}}$ *and so that the rest coordinates of vectors* y *and* y' *should satisfy antipodal conditions (7).*

The solutions, given in the theorem, correspond, obviously, to points of the cubic surface $w = n$.

From Theorem 2 it follows that the question about the existence of half-circulant Hadamard matrices of order $4n$, and consequently the existence of a regular hypersimplex inscribed into the $(4n-1)$-dimensional cube, reduces to the question about the solvability in the real number of a system of quadratic equations, in which the number of equations (which equals $5n-3$) exceed the number of unknown quantities $(4n-2)$. Thus, the existence of even one solution is not guaranteed. Therefore we modify it so that these numbers should coincide, that will provide the existence of complex solutions and, eventually, the existence of real solutions of the modified system under some additional conditions.

To this end let us introduce by analogy with $w = w(y)$ another polynomial $w' = {w'_0}^3 + \sum_{m=1}^{n-1}({w'_m}^3 + {w'_{2n-1-m}}^3)$, where $w'_i = \sum_{k=0}^{2n-2} t_{ik} y'_k$, and consider the following system of $4n-2$ equations:

$$\begin{cases} \frac{\partial w}{\partial y_i} \;=\; 3y_i, \qquad i = 1, 2, \ldots, 2n-2, \\ (y_0 + \sqrt{2}\sum\limits_{j=1}^{n-1} y_j)^2 (y_0 + \sqrt{2}\sum\limits_{j=1}^{n-1} y_j - \sqrt{2n-1})^2 + (y'_0 - \frac{n}{\sqrt{2n-1}})^4 \\ + \sum\limits_{j=1}^{n-1} (y_j^2 + y_{2n-1-j}^2 + {y'_j}^2 + {y'}_{2n-1-j}^2 - \frac{2n}{2n-1})^2 = 0, \\ \frac{\partial w'}{\partial y'_i} = 3y'_i, \qquad i = 0, 1, \ldots, 2n-2. \end{cases} \tag{8}$$

This system consists from two subsystems similar to (4), namely, one is in terms of the vector $y = \{y_0, y_1, \ldots, y_{2n-2}\}$ (first $2n-2$ equations) and another is in terms of the vector $y' = \{y'_0, y'_1, \ldots, y'_{2n-2}\}$ (last $2n-1$ equations), and also common for its the middle equation, which takes into account antipodal conditions (7). In [1, p. 66] the following existence theorem was announced.

Theorem 3. *The system of* $4n-2$ *nonhomogeneous algebraic equations with* $4n-2$ *unknown quantities* $y_0, y'_0, \ldots, y_{2n-2}, y'_{2n-2}$ *(8) has for any integer* $n > 1$ 2^{4n-1} *solutions, with regard their multiplicity. If* y_0 *takes a real value at least for one of this solutions, then a half-circulant Hadamard matrix of order* $4n$ *exists and into the* $(4n-1)$*-dimensional cube one can inscribe a regular simplex of the same dimension.*

2. Proof of the existence theorem

Let us transform system (8) to the homogeneous one by two sets of variables with the aid of additional coordinates y_{2n-1} and y'_{2n-1} (with regard that w and w', by definition, are homogeneous polynomials of third degree).

$$\begin{cases} \frac{\partial w}{\partial y_i} = 3y_i y_{2n-1}, \quad i = 1, 2, \ldots, 2n-2, \\ (y_0 + \sqrt{2}\sum\limits_{j=1}^{n-1} y_j)^2 \cdot (y_0 + \sqrt{2}\sum\limits_{j=1}^{n-1} y_j - \sqrt{2n-1}\,y_{2n-1})^2 {y'}_{2n-1}^4 \\ +(y'_0 - \frac{n}{\sqrt{2n-1}} y'_{2n-1})^4 y_{2n-1}^4 + \sum\limits_{j=1}^{n-1} [(y_j^2 + y_{2n-1-j}^2) {y'}_{2n-1}^2 \\ + ({y'_j}^2 + {y'}_{2n-1-j}^2) y_{2n-1}^2 - \frac{2n}{2n-1}\, y_{2n-1}^2 {y'}_{2n-1}^2]^2 = 0, \\ \frac{\partial w'}{\partial y'_i} = 3y'_i y'_{2n-1}, \quad i = 0, 1, \ldots, 2n-2. \end{cases} \tag{9}$$

Actually, every equation of system (9) is homogeneous both with respect to variables $y_0, y_1, \ldots, y_{2n-1}$ and also variables $y'_0, y'_1, \ldots, y'_{2n-1}$, and besides the first $2n-2$ equtions are homogeneous of second degree with respect to the first set of variables and zero degree with respect to the second ones, but for the last $2n-1$

equations opposite property is valid. Therefore one can consider system (9) on the product of two projective spaces $P^{2n-1} \times P'^{\,2n-1}(y_0, \ldots, y_{2n-1}, y'_0, \ldots, y'_{2n-1})$. By well-known Bezout's theorem (see [4, p. 268–269]) this system of $4n-2$ equations with $4n$ unknown quantities has 2^{4n-1} solutions with regard their multiplicity (if the number of its solutions is finite, what will follow from the present proof), that equals the intersection index

$$\sum k_{i_1} \cdots k_{i_{2n-1}} k'_{j_1} \cdots k'_{j_{2n-1}} = 2^{2n} \cdot 2^{2n-1} = 2^{4n-1},$$

where k_m, k'_m are the homogeneous degrees of the equation with number m in (9) with respect to coordinates in P^{2n-1} or $P'^{\,2n-1}$ respectively and the sum is over all permutations $(i_1 \ldots i_{2n-1} j_1 \ldots j_{2n-1})$ of integers $1, 2, \ldots 4n-2$ with $i_1 < i_2 < \cdots < i_{2n-1}$; $j_1 < j_2 < \cdots < j_{2n-1}$, where a nonzero contribution gives only one summand, with $i_{2n-1} = 2n-1$. Underline that only nonzero solutions are taken into account, i.e., the solutions in which are not vanish all coordinates into P^{2n-1} or those into $P'^{\,2n-1}$, and all proportional one another solutions are taken just the same solution [4, p. 272].

First, let us prove that for any nontrivial solution of homogeneous system (9) both y_{2n-1} and y'_{2n-1} do not vanish, i.e., every its solution induces the corresponding one of nonhomogeneous system (8).

Actually, it follows from the smoothness of the cubic surface $w' = n$ in $P'^{\,2n-1}$ (see Introduction) that $y'_{2n-1} \neq 0$ (in the opposite case, all $\frac{\partial w'}{\partial y'_i}$ are vanishing according to (9), and consequently there are singular points on this hypersurface). Therefore one can assume $y'_{2n-1} = 1$. Then last $2n-1$ equations of system (9) coincide with corresponding equations of system (8), which form a system of type (4). By virtue of its idempotency, we have $w_i'^{\,2} = \sum_{k=0}^{2n-2} t_{ik} y'_k = w'_i$, $i = 0, 1, 2, \ldots, 2n-2$. Thus, this subsystem of system (8) has, as the system of type (4), exactly 2^{2n-1} real solutions, and this equals the degree product of corresponding equations of (9).

The remaining $2n-1$ equations of system (9) one can consider as a system in P^{2n-1}, if in its last equation $y'_0, y'_1, \ldots, y'_{2n-2}, y'_{2n-1}$ will be replaced by any solution of the lower subsystem of system (9) with $y'_{2n-1} = 1$.

Let us assume that $y_{2n-1} = 0$ for a nontrivial solution of the considered system in P^{2n-1}. Then we obtain

$$\begin{cases} \frac{\partial w}{\partial y_i} = 0, \qquad i = 1, 2, \ldots, 2n-2, \\ (y_0 + \sqrt{2} \sum\limits_{j=1}^{n-1} y_j)^4 + \sum\limits_{j=1}^{n-1} (y_j^2 + y_{2n-1-j}^2)^2 = 0. \end{cases} \tag{10}$$

Introduce in this system one more equation (number one): $\frac{\partial w}{\partial y_0} = 3t^2\sqrt{2n-1}$, where t is an unknown parameter, whose value will determine later. All these equations, except the last, form a system of type (4) with the vector

$y = \{t^2\sqrt{2n-1}, 0, \ldots, 0\}$. We have in view of the idempotent property of system (4): $w_i^2 = \sum_{k=0}^{2n-2} t_{ik} \cdot 0 + t_{i0} \cdot t^2\sqrt{2n-1} = t^2$, i.e., for any admissible value of index i $w_i = \pm t$. Let us assume that $x_i = \bar{w_i}$, where $\bar{w_i}$ is one of admissible values of w_i. Then we obtain from (5) $y_i = c_i t$, $i = 0, 1, \ldots, 2n-2$, where all coefficients c_i are real numbers and besides $c_0 \neq 0$ (since any sum of $2n-1$ summands, which equals 1 or -1, obviously, cannot vanish). Substituting these values y_i into the last equation of system (10), we have

$$t^4[(c_0 + \sqrt{2}\sum_{j=1}^{n-1} c_j)^4 + \sum_{j=1}^{n-1}(c_j^2 + c_{2n-1-j}^2)^2] = 0,$$

hence $t = 0$, and so $y_0 = y_1 = \ldots = y_{2n-2} = 0$. Since by assumption $y_{2n-1} = 0$ too, then the given solution is trivial. Obtained contradiction proves that $y_{2n-1} \neq 0$, and consequently one can assume that $y_{2n-1} = 1$. With regard that $y'_{2n-1} = 1$ too, every solution of homogeneous system (9) induces the corresponding solution of nonhomogeneous system (8), i.e., the last one has 2^{4n-1} solutions. This is the first to be proved.

Let us assume $z = y_0 + \sqrt{2}\sum_{j=1}^{n-1} y_j$ and prove that all solutions of the upper subsystem of system (8) satisfy the following relation:

$$\frac{\partial w}{\partial y_0} = 3(y_0 - z + \frac{z^2}{\sqrt{2n-1}}). \tag{11}$$

In fact, summing termwise the first $n-1$ equations of system (8) and differetiating the polynomial w, we obtain

$$\begin{aligned} 3\sum_{j=1}^{n-1} y_j &= 3w_0^2\frac{\sqrt{2}(n-1)}{\sqrt{2n-1}} + \sum_{m=1}^{n-1}[3(w_m^2 + w_{2n-1-m}^2)\frac{\sqrt{2}}{\sqrt{2n-1}}\sum_{j=1}^{n-1}\cos\frac{2\pi mj}{2n-1}] \\ &= \frac{3\sqrt{2}}{\sqrt{2n-1}}[(n-\frac{1}{2})w_0^2 - \frac{1}{2}(w_0^2 + \sum_{m=1}^{n-1}(w_m^2 + w_{2n-1-m}^2))] \\ &= \frac{1}{\sqrt{2}}[\frac{3}{\sqrt{2n-1}}(y_0 + \sqrt{2}\sum_{j=1}^{n-1} y_j)^2 - \frac{\partial w}{\partial y_0}]. \end{aligned}$$

After transfering $\frac{\partial w}{\partial y_0}$ to the left hand side of obtained equality, we have

$$\frac{\partial w}{\partial y_0} + 3\sqrt{2}\sum_{k=1}^{n-1} y_j = \frac{3}{\sqrt{2n-1}}(y_0 + \sqrt{2}\sum_{j=1}^{n-1} y_j)^2 = \frac{3z^2}{\sqrt{2n-1}},$$

and relation (11) follows.

Adding equality (11) with a parameter z to system (8), we obtain the subsystem of $2n-1$ equations of type (4) with the vector $y = \{y_0 - z + \frac{z^2}{\sqrt{2n-1}}, y_1, \dots, y_{2n-2}\}$. By the idempotent property for all $i = 0, 1, 2, \dots, 2n-2$ we have

$$w_i^2 = \sum_{k=0}^{2n-2} t_{ik} y_k \; + \frac{1}{\sqrt{2n-1}}(-z + \frac{z^2}{\sqrt{2n-1}}) = w_i - \frac{z}{\sqrt{2n-1}} + \frac{z^2}{2n-1}\,.$$

This quadratic equation has following roots

$$2w_i = 1 \pm (1 - \frac{2z}{\sqrt{2n-1}}).$$

Consequently, we obtain for every admissible i: $w_i = \frac{z}{\sqrt{2n-1}}$ or $w_i = 1 - \frac{z}{\sqrt{2n-1}}$. Hence, if z is a complex number, then the imaginary parts of w_i and w_m at $m \neq i$ may differ only by sign. Besides, assuming in (5) $x_i = w_i$, we see, that for all solutions of our system $\sum_{i=0}^{2n-2} w_i = \sqrt{2n-1}\, y_0$ holds. Since by conditions of our theorem y_0 is a real number, it follows that all w_i are real too (because the sum of odd number of complex summands whose imiginary parts have equal absolute values cannot be a real number). It follows then, that z is a real number, if y_0 is real, and consequently $y_1, y_2, \dots, y_{2n-2}$ are real too. And by the proved earlier unknown quantities $y'_0, y'_1, \dots, y'_{2n-2}$ may take so only real values. Thus all summands of the middle eqution of system (8) should vanish, as nonnegative quantities.

From the vanishing of the first summand in this equation it follows that $z = y_0 + \sqrt{2}\sum_{j=1}^{n-1} y_j = 0$ or $z = \sqrt{2n-1}$. Then from the relation (11) in any case we have $\frac{\partial w}{\partial y_0} = 3y_0$ and our solution satisfies the first equation of system (4), which is obtained at $i = 0$. Therefore coordinates of vector y and, by the proved above, also coordinates of vector y' satisfy all the equtions of system (4). Moreover, y and y' satisfy antipodal conditions (7), what follows from the vanishing of the last summand of the middle equation of system (8). Next, from the vanishing of its second summand, it follows that $y'_0 = \frac{n}{\sqrt{2n-1}}$, i.e., the vectors y и y' satisfy all the conditions of Theorem 2, except the last one for y_0. Let us show that this condition for y_0 may be assumed also satisfied.

Indeed, since $\frac{\partial w}{\partial y_0} = 3y_0$, we obtain the following quadratic equation for y_0 from equality (6):

$$y_0^2 - \sqrt{2n-1}y_0 + \sum_{i=1}^{2n-2} y_i^2 = 0. \tag{12}$$

Similar equality is true for $y'_0, y'_1, \dots, y'_{2n-2}$ (it goes in the lower subsystem of (8) with $i = 0$). Since $y'_0 = \frac{n}{\sqrt{2n-1}}$, then $\sum_{i=1}^{2n-2} {y'_i}^2 = \sqrt{2n-1}y'_0 - {y'_0}^2 = \frac{n(n-1)}{2n-1}$.

Summing termwise antipodal conditions (7), which are valid for our real-valued solution, we have

$$\sum_{i=1}^{2n-2} y_i^2 = \frac{2n(n-1)}{2n-1} - \sum_{i=1}^{2n-2} {y'_i}^2 = \frac{n(n-1)}{2n-1}.$$

Substituting this in equation (12), we determine admissible values for y_0: $y_0 = \frac{n}{\sqrt{2n-1}}$ or $y_0 = \frac{n-1}{\sqrt{2n-1}}$. But in case $y_0 = \frac{n-1}{\sqrt{2n-1}}$, there is for the conjugate solution $\bar{y} = \{\sqrt{2n-1} - y_0, -y_1, \ldots, -y_{2n-2}\}$ by Lemma 1: $\bar{y}_0 = \sqrt{2n-1} - \frac{n-1}{\sqrt{2n-1}} = \frac{n}{\sqrt{2n-1}}$. So that if $y_0 \neq \frac{n}{\sqrt{2n-1}}$, one can replace a given solution by the conjugate one. And then it will satisfy the middle equation of system (8) together with $y' = \{y'_0, y'_1, \ldots, y'_{2n-2}\}$, since its first summand vanish too by Lemma 2 (at $\alpha_i = \sqrt{2n-1}$ for all $i = 1, 2, \ldots, n-1$), and besides the rest summands do not change at all their forms. Consequently, without loss of generility we may assume that for our real-valued solution $y_0 = \frac{n}{\sqrt{2n-1}}$.

Thus vectors $y = \{y_0, y_1, \ldots, y_{2n-1}\}$ and $y' = \{y'_0, y'_1, \ldots, y'_{2n-2}\}$, inducing together the given real solution of system (8), satisfy all conditions of Theorem 2. Therefore we obtain the existence of a half-circulant Hadamard matrix of order $4n$ hence, and that is, of a regular hypersimplex inscribed into the $(4n-1)$-dimensional cube as well. This concludes the proof.

Note that for construction of Hadamard's matrix of order $4n$, whose existence follows from Theorem 3, it is necessary to assume in (2) $x_{|m-i|} = w_{|m-i|}$, $x'_{|m+i-2|} = w'_{|m+i-2|}$, where w_s and w'_s are obtained from (3) by substitution $y_0 = \frac{n}{\sqrt{2n-1}}, y_1, \ldots, y_{2n-2}$ and $y'_0 = \frac{n}{\sqrt{2n-1}}, y'_1, \ldots, y'_{2n-2}$ in its. The coordinates of all vertices of a regular hypersimplex inscribed into the $(4n-1)$-dimensional cube answer to rows of the matrix $\bar{H}$, which is obtained by removing from Hadamard's matrix H its first column.

R e m a r k 1. It follows from our proof of the existence theorem that its assertion holds if the first summand in the middle equation of system (8) is replaced by the following one: $(y_0 + \sqrt{2}\sum_{j=1}^{n-1} y_j)^4$ or $(y_0 + \sqrt{2}\sum_{j=1}^{n-1} y_j - \sqrt{2n-1})^4$, since antipodal conditions (7) are correct for vectors $\bar{y} = \{\sqrt{2n-1} - y_0, -y_1, \ldots, -y_{2n-2}\}$ and y' as soon as they are true for vectors $y = \{y_0, y_1, \ldots, y_{2n-2}\}$ and y', because at $i \geq 1$ for the conjugate solution $\bar{y}_i^2 = (-y_i)^2 = y_i^2$. This is directly related with the existence theorem announced in [1].

As we have seen, unknown quantity y_0 plays a special role in Theorem 3. But this is not quite the right.

In fact, as it was determined in the proof of Theorem 3, for any admissible value $i = 0, 1, 2, \ldots, 2n-2$ $\quad w_i = \frac{z}{\sqrt{2n-1}}$ or $w_i = 1 - \frac{z}{\sqrt{2n-1}}$, where $z = y_0 + \sqrt{2}\sum_{j=1}^{n-1} y_j$. So that $w_i - \frac{1}{2} = \pm(\frac{z}{\sqrt{2n-1}} - \frac{1}{2}) = \pm\Delta$, where $\Delta = \frac{z}{\sqrt{2n-1}} - \frac{1}{2}$. At the same time from equalities (5) it follows, with regard of $x_k = w_k(y_0, y_1, \dots, y_{2n-2})$ and the identity $\frac{1}{2} + \sum_{m=1}^{n-1} \cos \frac{2\pi jm}{2n-1} \equiv 0$, that

$$\begin{cases} y_0 = \frac{1}{\sqrt{2n-1}} \sum\limits_{i=0}^{2n-2} [(w_i - \frac{1}{2}) + \frac{1}{2}] = \frac{\sqrt{2n-1}}{2} + \Delta c_0, \\ y_j = \sqrt{\frac{2}{2n-1}} [w_0 - \frac{1}{2} + \sum\limits_{m=1}^{n-1} [(w_m - \frac{1}{2}) + (w_{2n-1-m} - \frac{1}{2})] \cos \frac{2\pi jm}{2n-1}] = \Delta c_j, \\ y_{2n-1-j} = \sqrt{\frac{2}{2n-1}} \sum\limits_{m=1}^{n-1} [(w_m - \frac{1}{2}) - (w_{2n-1-m} - \frac{1}{2})] \sin \frac{2\pi jm}{2n-1} = \Delta c_{2n-1-j}, \end{cases} \tag{13}$$

where c_0, c_j, c_{2n-1-j} are some real numbers, depending on the signs of differences $w_k - \frac{1}{2}$ for $k = 0, 1, 2, \dots, 2n-2$, which are determined by the solution of system (8) itself. Consequently, if at least one of $y_i \neq 0$ at $i \geq 1$ is a real number, then all of the rest unknown quantities $y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_{2n-2}$ are real too, since $\Delta = \frac{z}{\sqrt{2n-1}} - \frac{1}{2}$, that is z, will be then real numbers.

Theorem 4. *The assertion of Theorem 3 holds if the condition for y_0 to be real is replaced by that for any other $y_i \neq 0$, $1 \leq i \leq 2n-2$.*

R e m a r k 2. Strictly speaking, homogeneous system (9), using in the proof of the existence theorem, has exactly 2^{4n-1} solutions, if it is in general case. Otherwise, the number of its solutions is infinite. But it follows from relations (13) that this number is finite, since the number Δ must satisfy an equation of fourth degree, which arises from the middle equation of system (8) after the substitution all y_i from (13). So that system (9) is in general case and has by Bezout's theorem exactly 2^{4n-1} solutions.

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