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Retroreflecting Curves in Nonstandard Analysis

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We present a direct construction of retroreflecting curves by means of Nonstandard Analysis. We construct non self-intersecting curves which are of class C^1, except for a hyper-finite set of values, such that the probability of a particle being reflected from the curve with the velocity opposite to the velocity of incidence, is infinitely close to 1. The constructed curves are of two kinds: a curve infinitely close to a straight line and a curve infinitely close to the boundary of a bounded convex set. We shall see that the latter curve is a solution of the problem: find the curve of maximum resistance infinitely close to a given curve.

Key words: Nonstandard Analysis, retroreflectors, maximum resistance problems, reflection, billiards.

Mathematics Subject Classification 2000: 26E35, 49K30, 49Q10.

1. Introduction

A. A *retroreflector* is an optical device that sends incident beams of light back to their origin. If the retroreflector is much smaller than the size of the source of light, it actually reverses the direction of light. We proceed to define a mathematical retroreflector.

Consider a set with piecewise smooth boundary, and the billiard in the complement of this set. The set is called *mathematical retroreflector* (or just retroreflector), if almost all incident particles are reflected in such a way that the velocity of reflection is opposite to the velocity of incidence. In this paper we shall construct two-dimensional retroreflectors by means of Nonstandard Analysis.

As far as we know, it is the first time that nonstandard analysis techniques are used within the framework of mathematical retroreflectors theory. In [6], an *asymptotically retroreflecting* sequence of sets was constructed. More precisely, the sets in the sequence presented in [6] are contained in one fixed bounded convex set and contain another one. "Asymptotically retroreflecting" means that the sum of the incidence velocity and the reflection velocity converges in measure to zero, with both the velocities being considered as functions on the (measurable) set of all incident particles. In [5], an asymptotically retroreflecting sequence of unbounded sets was constructed, each of them containing a fixed half-plane and contained in another one.

One can easily construct a *partial* retroreflector; from Fig. 1, one can see that only a part of the incident particles is reversed.

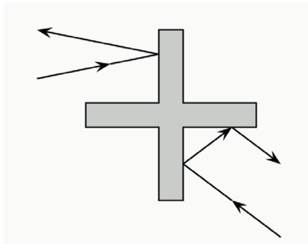

Fig. 1: A partial retroreflector.

B. Let us formulate the main results of the paper. First, consider a set Ω with piecewise smooth boundary, contained in the lower half-plane,

$$\Omega \subset \{(x, y) \mid \ y \leq 0\} \subset \mathbb{R}^2$$

and define the mapping $(\xi, \theta) \mapsto \theta^+_\Omega(\xi, \theta)$ as follows (see Fig. 2).

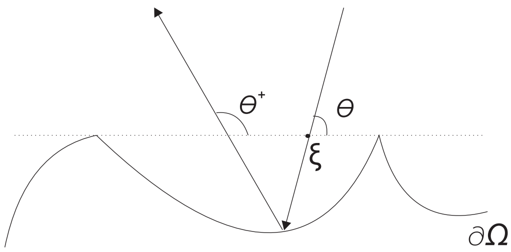

Fig. 2: Angle of reflection.

Consider the billiard in $\mathbb{R}^2 \setminus \Omega$. Tag billiard particles incident on Ω by their point of the first intersection with the straight line $y = 0$ and by the velocity at the moment of intersection. That is, let a particle intersect the line at the point $(\xi, 0)$ and let the velocity at this point be $v = -(\cos\theta, \sin\theta)$; then tag this particle by $(\xi, \theta) \in \mathbb{R} \times [0,\, \pi]$. The particle makes several reflections from $\partial\Omega$ and finally intersects the line $y = 0$ again and moves freely afterwards. Denote the final velocity by $v^+ = (\cos\theta_\Omega^+(\xi,\theta), \sin\theta_\Omega^+(\xi,\theta))$. The mapping $(\xi,\theta) \mapsto \theta_\Omega^+(\xi,\theta)$ is defined on a subset of $\mathbb{R} \times [0,\, \pi]$.

Theorem 1. *There exists Ω such that its boundary $\partial\Omega$ is a nonselfintersecting curve infinitely close to the line $y = 0$ and invariant with respect to the shift $(x, y) \mapsto (x+1, y)$. Moreover, for all $(\xi,\theta) \in [0,\, 1] \times [0,\, \pi]$, $\theta_\Omega^+(\xi,\theta) - \theta \approx 0$ holds, except for a set of measure ≈ 0.*

Theorem 1 means that nearly all incident particles almost reverse direction, and the reflecting set is obtained from the half-plane by an infinitely small modification near its boundary.

C. Now fix a convex bounded set $B \subset \mathbb{R}^2$ with nonempty interior and consider a set $\Lambda \subset B$ with piecewise smooth boundary $\partial\Lambda$. Define the mapping $(\xi,\theta) \mapsto \theta_\Lambda^+(\xi,\theta)$ in a similar way. Namely, consider the billiard in $\mathbb{R}^2 \setminus \Lambda$. Let an incident particle intersect ∂B for the first time at the point ξ and let the velocity at this point form the angle θ with the tangent to ∂B at ξ. The particle makes several reflections from Λ, then intersects ∂B again and finally moves freely, the final velocity making the angle $\theta_\Lambda^+(\xi,\theta)$ with the tangent.

The mapping θ_Λ^+ is defined on a subset of $\partial B \times [0,\, \pi]$.

Theorem 2. *There exists a set Λ_* such that the boundary $\partial\Lambda_*$ is a closed nonselfintersecting curve infinitely close to ∂B and such that for all $(\xi,\theta) \in \partial B \times [0,\, \pi]$, $\theta_{\Lambda_*}^+(\xi,\theta) - \theta \approx 0$ holds, except for a set of measure ≈ 0.*

D. There is an application of these results in Newtonian aerodynamics. Suppose that a body Λ moves forward through a highly rarefied medium, and at the same time slowly rotates. Due to elastic collisions between the body and the medium particles, a braking force acting on the body in the direction opposite to its motion is created. This force is called the force of aerodynamic resistance, or just *resistance*.

The mean value of resistance is given by the formula

$$R(\Lambda) = \frac{3}{8} \int_{\partial B} \int_0^\pi \big(1 + \cos(\theta_\Lambda^+(\xi,\theta) - \theta)\big) \sin\theta \, d\theta \, d\xi; \tag{1}$$

the factor $3/8$ is chosen in such a way that substituting $\Lambda = B$ one gets $R(B) = |\partial B|$, that is, resistance of the convex set B is just its perimeter.

Consider the problem: maximize $R(\Lambda)$ over all sets $\Lambda \subset B$ such that $\partial\Lambda$ is near ∂B. The solution is given, say dynamically, by the sets Λ_* determined in Th. 2, for which $R(\Lambda_*) \approx 1.5$.

The paper is organized as follows. Theorems 1 and 2 are proved in Sects. 3 and 4, respectively. In Section 5 the maximization problem is examined in more detail.

2. Self-Intersecting Mirrors

We present a rather elementary direct approach to this problem by means of (nonstandard) Infinitesimal Calculus. As in [5], we use the basic reflection property of the ellipse (Fig. 3):

rays which hit between the foci are also reflected between the foci.

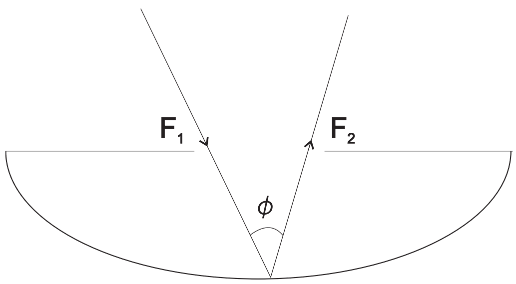

Fig. 3: Reflection in an ellipse.

In particular, if the ellipse has foci $F_1(-c, 0)$, $F_2(c, 0)$, equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and eccentricity $c/a \approx 0$, then the angle of reflection ϕ is infinitesimal, *i.e.*, reflection is almost opposite to incidence.

Assume light rays may have any direction whatsoever from above a line segment of length 1 and fix internal sequences $M_i,\ N_i \in {}^*\mathbb{N}_\infty$ for $i \in {}^*\mathbb{N}$ (where ${}^*\mathbb{N}_\infty$ denotes the set of infinite hypernatural numbers).

Divide the segment $[0, 1]$ in N_1 equal parts and in each of them define an ellipse with the major axis on the initial segment, as shown in Fig. 4, where $F_{2i-1,1}$ and $F_{2i,1}$ denote the foci of the $i - th$ ellipse, $i = 1, \ldots, N_1$.

Each of the N_1 ellipses verifies the following conditions for exactness of subdivision:

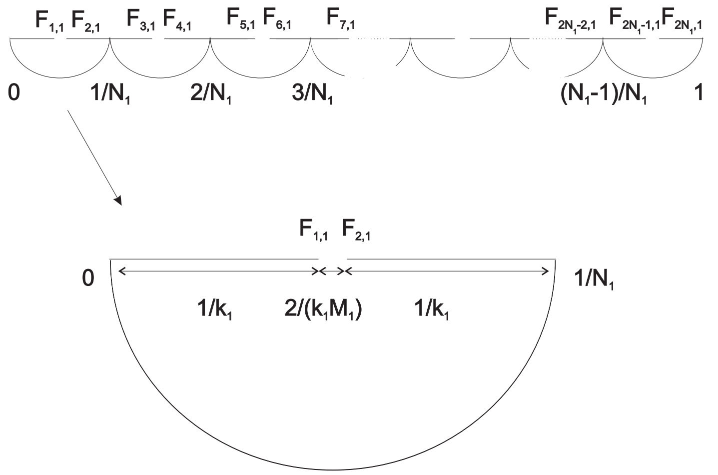

Fig. 4: First step.

$$k_1 = 2N_1 \left(1 + \frac{1}{M_1}\right),$$

$$a_1 = \frac{1}{k_1} + \frac{1}{k_1 M_1}, \ b_1 = \frac{1}{k_1}\sqrt{1 + \frac{2}{M_1}}, \ c_1 = \frac{1}{k_1 M_1}.$$

Therefore the eccentricity $e_1 \approx 0$ as required; but the probability P_1 that a light ray falls out of the foci window is given by

$$P_1 = N_1 \frac{2}{k_1} = \frac{M_1}{M_1 + 1} \approx 1.$$

Next define new ellipses in each of the segments $[(j-1)/N_1, F_{2j-1,1}]$ and $[F_{2j,1}, j/N_1]$ for $j = 1, \ldots, N_1$. Note that both segments have the length $1/k_1$ and divide each of them into N_2 equal parts wherein ellipses are defined again with foci $F_{2i-1,2}$ and $F_{2i,2}$, $i = 1, \ldots, N_2$, according to the following conditions:

$$k_2 = 2^2 N_1 N_2 \left(1 + \frac{1}{M_1}\right)\left(1 + \frac{1}{M_2}\right),$$

$$a_2 = \frac{1}{k_2} + \frac{1}{k_2 M_2}, \quad b_2 = \frac{1}{k_2}\sqrt{1 + \frac{2}{M_2}}, \quad c_2 = \frac{1}{k_2 M_2}.$$

The probability P_2 that a light ray falls out of the foci windows is given by

$$P_2 = 2N_1 N_2 \frac{2}{k_2} = \left(\frac{M_1}{M_1 + 1}\right)\left(\frac{M_2}{M_2 + 1}\right).$$

Iteration of this procedure follows the pattern

$$k_i = 2^i \prod_{j=1}^{i} N_j \prod_{j=1}^{i} \left(1 + \frac{1}{M_j}\right),$$

$$a_i = \frac{1}{k_i} + \frac{1}{k_i M_i}, \quad b_i = \frac{1}{k_i}\sqrt{1 + \frac{2}{M_i}}, \quad c_i = \frac{1}{k_i M_i}.$$

Interestingly enough, whatever the sequence N_i might be

$$P_i = \prod_{j=1}^{i} \frac{M_j}{M_j + 1}.$$

In particular, if for some fixed $N \in {}^*\mathbb{N}_\infty$ all the $M_j = N$, then

$$P_{N^2} = \left(1 - \frac{1}{N+1}\right)^{N^2} \approx e^{-\frac{N^2}{N+1}} \approx 0. \tag{2}$$

Assume from now on that for some fixed $N \in {}^*\mathbb{N}_\infty$, $M_j \equiv N$ so that (2) holds.

The possibility that a ray entering a foci window hits one of the smaller ellipses and is not reflected conveniently must also be considered. The following discusses this situation. Consider Fig. 5, where one ellipse is centered at the origin of coordinates for simplicity.

Let the light ray r pass through the window $[F_{1,i-1}F_{2,i-1}]$ with inclination θ.

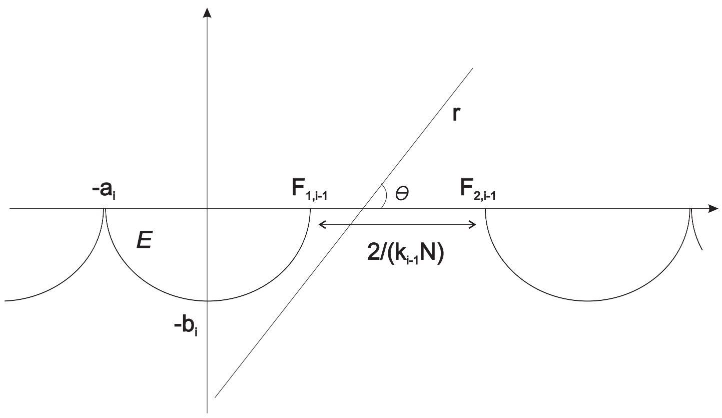

Fig. 5: Avoiding inconvenient hits.

As a matter of notational simplification, define

$$A := 2K_{i-1}N_i \quad \text{and} \quad B := \frac{K_{i-1}N}{2}.$$

The centered ellipse is given by

$$\frac{x^2}{a_i^2} + \frac{y^2}{b_i^2} = 1 \ \text{ with } \ a_i = \frac{1}{A} \ \text{and} \ b_i = \frac{N}{A(N+1)}\sqrt{1+\frac{2}{N}}.$$

An equation of the light ray is

$$y_t = \tan\theta\left(x - \frac{1}{A} - \frac{t}{B}\right) \ \text{for some } t \in]0,1[.$$

The light ray intersects the ellipse at a point (x, y_t) when

$$\theta = \arctan\left(\frac{\sqrt{1-x^2A^2}}{B+At-ABx}\cdot B\cdot\frac{\sqrt{N(N+2)}}{N+1}\right)$$

necessarily with

$$0 < x < 1/A;$$

but then

$$\begin{aligned} 0<\theta &< \arctan\left(\frac{B}{At}\cdot\frac{\sqrt{N(N+2)}}{N+1}\right) \\ &= \arctan\left(\frac{N}{N_it}\cdot\frac{\sqrt{N(N+2)}}{4(N+1)}\right) \end{aligned}$$

therefore $\theta \approx 0$ as long as $\frac{N}{N_it} \approx 0$ and this happens whenever $t \geq \frac{1}{N}$ and $N_i = N^3$, thus the probability that the entering light rays hit a smaller ellipse is approximately

$$\begin{aligned} &\sum_{j=1}^{N^2-1}\frac{2^{j+1}}{N^2k_j}\prod_{i=1}^{j}N_i = \frac{2}{N^2}\sum_{j=1}^{N^2-1}\left(\frac{N}{N+1}\right)^j \\ &= \frac{2}{N}\left(1-\left(\frac{N}{N+1}\right)^{N^2-1}\right) \approx \frac{2}{N}\left(1-e^{-\frac{N^2-1}{N+1}}\right) \end{aligned}$$

hence infinitesimal. Summarizing:

> *As long as all the $M_i = N$ and $N_i = N^3$, for some $N \in {}^*\mathbb{N}_\infty$, the N^2-th step of the foregoing procedure entails a selfintersecting "mirror" which reflects light rays along lines infinitely near the incidence lines with probability infinitely near 1.*

Although selfintersecting, our curve is $^* -$ *continuous* and infinitely resistant.

3. Simple Mirrors

From now on we will take all the $N_i = N^3$.

We eliminate self-intersections "indirectly" as illustrated in Fig. 6: extend the mirror infinitesimally towards the center of each ellipse $[-c_i, -P] \cup [P, c_i]$, and connect with the ellipse itself by means of two straight line segments r and $\overline{r}$ of adequate inclination ϕ.

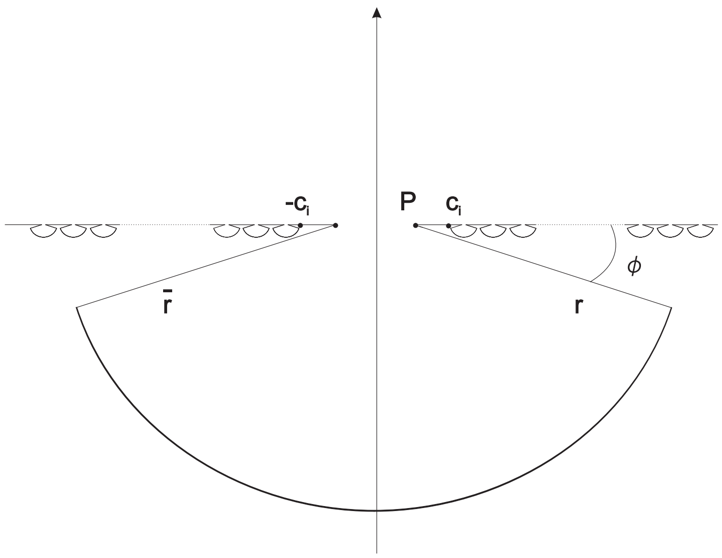

Fig. 6: Eliminating self-intersections.

The angle ϕ must of course be infinitesimal, but also such that the line r, and its symmetric $\overline{r}$, do not intersect any of the inner ellipses. Finally, having thus created more "reflective" regions, their total length must be infinitesimal. We now sketch calculations

$$c_i = \frac{1}{k_i N}, \qquad a_{i+1} = \frac{1}{2k_i N^3},$$

$$b_{i+1} = \frac{1}{2(N+1)k_i N^2}\sqrt{1 + \frac{2}{N}}.$$

For some positive ϵ to be determined, the center C of the first inner ellipse and the end point P verify

$$C = c_i + a_{i+1} = \frac{2N^2+1}{2k_i N^3}, \qquad P = \frac{c_i}{1+\epsilon}.$$

The line r and inner ellipse $\mathcal{E}$ satisfy

$$r \equiv y = \tan\phi\,(x-P)\ , \qquad \mathcal{E} \equiv \frac{(x-C)^2}{a_{i+1}^2} + \frac{y^2}{b_{i+1}^2} = 1.$$

The angle τ for which r is tangent to $\mathcal{E}$ is given by

$$\tau = \arctan \frac{-\frac{b_{i+1}}{a_{i+1}}\sqrt{a_{i+1}^2-(x-C)^2}}{x-P}, \qquad c_i < x < C.$$

Now, $\tau \approx 0$ whenever $\dfrac{\sqrt{a_{i+1}^2-(x-C)^2}}{x-P} \approx 0$; but,

$$0 \ \leq \ \frac{\sqrt{a_{i+1}^2-(x-C)^2}}{x-P} \ \leq \ \frac{a_{i+1}}{x-P} \ \leq \ \frac{a_{i+1}}{c_i}\frac{1+\epsilon}{\epsilon} \ \leq \ \frac{1}{N^2\epsilon}$$

and $\tau \approx 0$ when $\epsilon = \frac{1}{N}$. Any infinitesimal angle $\phi > \tau$ may be used to eliminate the self-intersection. Moreover, as

$$c_i - P = \frac{1}{k_i N(N+1)} < \frac{1}{N}\frac{2}{k_i N}\,,$$

the probability of a ray being inadequately reflected by this procedure is infinitesimal.

Summarizing, the probability of a ray being reflected with opposite direction of incidence is given by

$$\widehat{P_{N^2}} \approx 1 - \left(e^{-\frac{N^2}{N+1}} + \frac{2}{N}\left(1 - e^{-\frac{N^2-1}{N+1}}\right)\right) \approx 1.$$

4. Convex Mirrors

As a matter of making terminology more precise, let $\sigma : {}^*[0,1] \to {}^*\mathbb{R}^2$ be the curve thus defined in Sect. 3.

When one wants to take into account the size and the position of the mirror, an affine transformation is in order: given distinct points P and Q in $\mathbb{R}^2$, let

$$\begin{aligned} (v_1, v_2) &:= Q - P, \\ M &:= \begin{bmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{bmatrix}, \\ \sigma_{PQ}(t) &:= P + M\sigma(t), \quad t \in {}^*[0,1]; \end{aligned}$$

σ_{PQ} describes the (simple plane) mirror positioned along $\overrightarrow{v}$, which we may re-parametrize in $I := [a, b],\ a < b$, by

$$\sigma_{PQ}^{I}(t) \ := \ \sigma_{PQ}\left(\frac{t-a}{b-a}\right), \qquad t \in I. \tag{3}$$

Suppose now that $\alpha : [0, \ell] \subseteq \mathbb{R} \to \mathbb{R}^2$ is a C^1 regular curve parameterized by arc length*. Let the "reflective side" of α be its convex side as illustrated in Fig. 7.

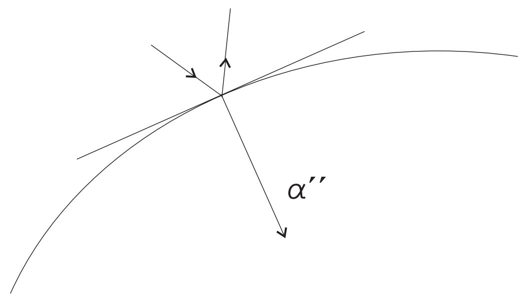

Fig. 7: Convex mirror.

A mirror of almost maximum resistance adjusted to the curve may be described in the following way

1. Pick an infinite $N \in {}^*\mathbb{N}_\infty$ and define for $0 \le j \le 2N$:

$$a_j \ := \ \begin{cases} \frac{j/2}{N}\,, & \text{if j is even,} \\ \frac{(j+1)/2}{N} - \frac{1}{N^2}\,, & \text{if j is odd,} \end{cases}$$

$$b_j \ := \ \ell a_j,$$

so that

$$[0, \ell] \ = \ \bigcup_{j=1}^{2N} [b_{j-1}, b_j],$$

$$b_j - b_{j-1} \ = \ \begin{cases} \frac{\ell}{N^2}\,, & j \text{ is even,} \\ \frac{\ell}{N} - \frac{\ell}{N^2}\,, & j \text{ is odd,} \end{cases} \quad 1 \le j \le 2N.$$

*Actually it suffices that α is rectifiable so that the following general procedure may be adapted.

2. Define

$$
\begin{aligned}
P_j &:= \alpha(b_j), & 0 \le j \le 2N,\\
I_j &:= [b_j, b_{j+1}], & 0 \le j \le 2N-1.
\end{aligned}
$$

and consider the polygon $[P_0, P_1, \ldots, P_{2N}]$. Also define, for $j \in \{0, \ldots, 2N-1\}$ (vide (3) above)

$$
\mu_j(t) := \begin{cases} \sigma^{I_j}_{P_j P_{j+1}}(t), & \text{if } t \in I_j \ \ \& \ j \text{ is even},\\[2ex] P_j + \dfrac{N^2}{\ell}(t-b_j)(P_{j+1}-P_j), & \text{if } t \in I_j \ \ \& \ j \text{ is odd}. \end{cases}
$$

Finally, $\mu_0 + \cdots + \mu_{2N-1}$ is a mirror of almost maximum resistance whose standard part is α. Under infinite magnification, the geometry between P_j and P_{j+2} with j even is exemplified in Fig. 8 below.

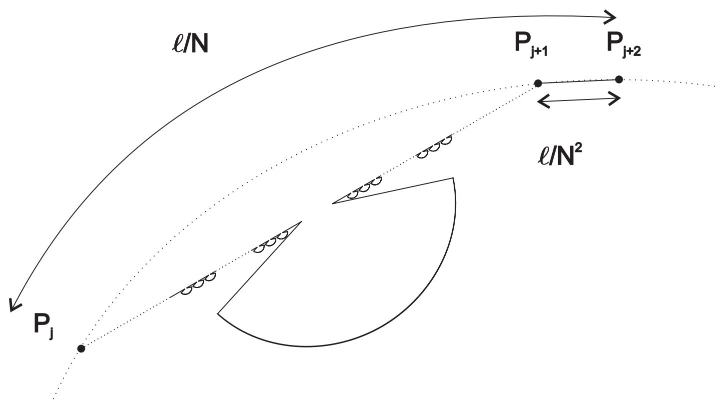

Fig. 8: Curve under infinitesimal microscope.

5. Calculus of the Resistance

We will now evaluate the resistance of the curve obtained in Sect. 3 by minimizing R. To do so, we must maximize the angle $\theta^+ - \theta$. We assume that the light ray hits one inner ellipse between the foci, so that the direction of the reflected ray is almost inverted (elsewhere the probability is approximately zero). Therefore the angle of reflection $\theta^+ - \theta$ is less than the angle of reflection when a ray light hits one of the foci (and consequently the ray is reflected to the second foci).

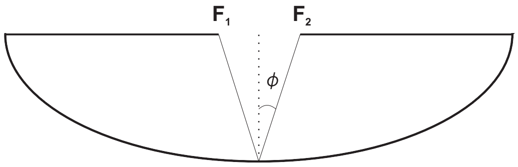

Fig. 9: Maximizing the angle of reflection.

Let us consider the general case (the i-step) and let ϕ be a half of the maximum angle of reflection, as exemplified in Fig. 9.

Therefore

$$\tan\phi = \frac{c_i}{b_i} = \frac{1}{\sqrt{N(N+2)}}$$

and so

$$\cos(\theta^+ - \theta) \geq \cos\left(2\arctan\frac{1}{\sqrt{N(N+2)}}\right) = 1 - \frac{2}{(N+1)^2}$$

and

$$R \gtrsim \frac{3}{8}\left(2 - \frac{2}{(N+1)^2}\right)\int_0^1\int_0^\pi \sin\theta\, d\theta\, d\xi = \frac{3}{4}\left(2 - \frac{2}{(N+1)^2}\right) \approx 1.5.$$

We also remark that the maximum resistance of any curve infinitely close to the segment $[0,1]\times\{0\}$ is 1.5.

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