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Bifurcations of Solitary Waves

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The paper provides a brief review of the recent results devoted to bifurcations of solitary waves. The main attention is paid to the universality of soliton behavior and stability of solitons while approaching supercritical bifurcations. Near the transition point from supercritical to subcritical bifurcations, the stability of two families of solitons is studied in the framework of the generalized nonlinear Schrödinger equation. It is shown that one-dimensional solitons corresponding to the family of supercritical bifurcations are stable in the Lyapunov sense. The solitons from the subcritical bifurcation branch are unstable. The development of this instability results in the collapse of solitons. Near the time of collapse, the pulse amplitude and its width exhibit a self-similar behavior with a small asymmetry in the pulse tails due to self-steepening.

Key words: stability, critical regimes, wave collapse.

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1. Introduction

According to the usual definition, solitons are nonlinear localized objects propagating uniformly with a constant velocity (see, for example, [1, 2]). The soliton velocity V represents the main soliton characteristics which often defines the soliton shape, in particular its amplitude and width.

It is well known that if the velocity V of a moving object is such that the equation

$$\omega_k = \mathbf{k} \cdot \mathbf{V} \tag{1}$$

has a nontrivial solution where $\omega = \omega_k$ is the dispersion law for linear waves and k is the wave vector, then this object will lose energy due to Cherenkov radiation. This also pertains, to a large extent, to solitons as localized stationary entities. They cannot exist if the resonance condition (1) is satisfied. Hence follows the first, and simplest, selection rule for solitons: the soliton velocity must be either less than the minimum phase velocity of linear waves or greater than the maximum phase velocity. The boundary separating the region of existence of solitons from the resonance region (1) determines the critical soliton velocity V_{cr}. As it is easily seen, this velocity coincides with the group velocity of linear waves at the touching point where the straight line $\omega = kV$ is tangent to the dispersion curve $\omega = \omega_k$ (in the multidimensional case — the point of tangency of the plane $\omega = \mathbf{k} \cdot \mathbf{V}$ to the dispersion surface). If touching occurs from below, then the critical velocity determines the maximum soliton velocity for this parameter range and, conversely, for touching from above V_{cr} coincides with the minimum phase velocity. Two regimes are possible in crossing this boundary corresponding to supercritical or subcritical bifurcations (soft or rigid excitation regimes).

While approaching the supercritical bifurcation point from below or above the soliton amplitude vanishes smoothly according to the same — Landau — law ($\propto |V - V_{cr}|^{1/2}$) as for the phase transitions of the second kind (see, for instance, [3]). The behavior of solitons in this case is completely universal, both for their amplitudes and their shapes. As $V \to V_{cr}$ solitons transform into oscillating wave trains with the carrying frequency corresponding to the extremal phase velocity of linear waves V_{cr}. The shape of the wave train envelope coincides with that for the soliton of the standard — cubic — nonlinear Schrödinger equation (NLS). The soliton width happens to be proportional to $|V - V_{cr}|^{-1/2}$.

Bifurcations of solitons were first observed for gravity-capillary waves in numerical simulations by Longuet–Higgins [4] and explained later in [5–9]. Then the bifurcation — a transition from periodic solutions to a soliton solution — was studied in [5] and [6] using normal forms. The stationary NLS for gravity-capillary wave solitons was derived in [8]. In [10] it was shown that this mechanism can be extended to optical solitons. In fact, this paper provided the first demonstration of the universality of soliton behavior near a supercritical bifurcation for waves of arbitrary nature. It is worth noting that the universal character of solitons allows not only to find their shapes, but also to study their stability. This analysis, as stated in [11], shows that near supercritical bifurcation the solitons are stable only in the one-dimensional case.

The question of whether the bifurcation is supercritical or subcritical depends on the character of nonlinear interaction. The supercritical bifurcation occurs for a focusing nonlinearity when the product $\omega'' T < 0$, where $\omega'' = \partial^2\omega/\partial k^2$ is the second derivative of the frequency with respect to the wave number, taken at the touching point $k = k_0$, and T is the value of the matrix element $T_{k_1 k_2 k_3 k_4}$ of the four-wave interaction for $k_i = k_0$. If $\omega'' T > 0$, which corresponds to a defocusing nonlinearity, then there are no solitons — localized solutions — with amplitude vanishing gradually as $V \to V_{cr}$. In the theory of phase transitions this corresponds to a first-order phase transition, and in the theory of turbulence, using Landau's terminology [12], it corresponds to a rigid regime of excitation. The transition through the critical velocity is accompanied by a jump in the soliton amplitude. The magnitude of the jump is determined by the next higher-order terms in the expansion of the Hamiltonian. Like for the first-order phase transitions, the universality of soliton behavior is no longer guaranteed in this situation. When the amplitude jump at this transition is small, it is enough to keep a finite number of next order terms in the Hamiltonian expansion to describe such a bifurcation. In the phase transitions this corresponds to a first-order phase transition close to a second-order transition, which occurs, for example, near a tri-critical point. As shown in [13], this situation arises for one-dimensional internal-wave solitons propagating along the interface between two ideal fluids with different densities in the presence of both gravity and capillarity. According to [13] the matrix element T in this case vanishes for density ratio $\rho_1/\rho_2 = (21 - 8\sqrt{5})/11$. This type of bifurcations can also be met for gravity water waves with finite depth when the matrix element $T = 0$ at $\theta_{cr} = k_0 h \approx 1.363$. In nonlinear optics, as shown in [11], a decrease of T (Kerr constant) can be provided by the interaction of light pulses with acoustic waves (Mandelstamm–Brillouin scattering). If the jump in soliton amplitude is of order one, then we need to keep all the remainder terms in the Hamiltonian expansion.

In this paper we give a brief review of the recent results devoted to this subject. The main attention will be paid to the universality of soliton behavior and stability of solitons while approaching the supercritical bifurcation point.

The paper is organized as follows. The next section is devoted to stationary solitons for arbitrary nonlinear wave media and their properties near the supercritical bifurcation. Section 2 deals with the stability of solitons based on the Lyapunov theorem and the Hamiltonian approach. It is shown by means of integral estimates of Sobolev type in their multiplicative variant (Gagliardo–Nirenberg inequalities) that only one-dimensional solitons are Lyapunov stable. It is worth noting that, in contrast to the method of normal forms, which is extensively used in [5, 6, 13, 14] for studying bifurcations of solitons, the Hamiltonian approach is fundamental for investigating soliton stability. In the method of normal forms, the introduction of envelopes is not unique. Consequently, the Hamiltonian equations of motion lose their initial Hamiltonian structure after their averaging. In this section it is shown that near the bifurcation point the multi-dimensional solitons are unstable due to the modulational instability. In the last section we consider which nonlinear effects must be taken into account near the transition from supercritical to subcritical bifurcations and how they change the shape of solitons and their stability.

2. Supercritical Bifurcations

Let us consider a purely conservative nonlinear wave medium which can be described by the Hamiltonian

$$H = \int \omega_k |a_k|^2 d\mathbf{k} + H_{int}, \tag{2}$$

where ω_k is the dispersion law of small-amplitude waves, a_k are normal amplitudes of the waves, and the Hamiltonian H_{int} describes the nonlinear interaction of the waves.

The equations of motion of the medium can be written in terms of the amplitudes a_k in the standard manner

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k = -i\frac{\delta H_{int}}{\delta a_k^*}, \tag{3}$$

so that in the absence of an interaction the system consists of a collection of noninteracting oscillators (waves):

$$a_k(t) = a_k(0)e^{-i\omega_k t}.$$

Equation (3) describes the dynamics in the wave number space. To go back to the physical space one needs to perform the inverse Fourier transform

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{d/2}} \int a_k(t) e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k}. \tag{4}$$

Originally, the function $\psi(\mathbf{x}, t)$ is related to the characteristics of the medium (fluctuations of the density and velocity of the medium, electric and magnetic fields, and so on) by a linear transformation (see, for example, [15]). It is important that if $\psi(\mathbf{x}, t)$ is a periodic function of the coordinates, then its spectrum $a_k(t)$ consists of a sum of δ-functions. For localized distributions $\psi(\mathbf{x}, t) \to 0$ as $|\mathbf{x}| \to \infty$. The Fourier amplitude $a_k(t)$, being a localized function of $\mathbf{k}$, does not contain δ-function singularities.

Let us now consider the solution of (3) in the form of a soliton propagating with the constant velocity $\mathbf{V}$:

$$\psi(\mathbf{x}, t) = \psi(\mathbf{x} - \mathbf{V}t).$$

In this case the entire dependence of a_k on time t is contained in the oscillating exponent:

$$a_k(t) = c_k e^{-i\mathbf{k}\cdot\mathbf{V}t},$$

where by virtue of (3) the amplitude c_k will satisfy the equation

$$(\omega_k - \mathbf{k}\cdot\mathbf{V})c_k = -\frac{\delta H}{\delta c_k^*} \equiv f_k. \tag{5}$$

The difference $\omega_k - \mathbf{k}\cdot\mathbf{V}$ appearing in this equation will be positive for all $\mathbf{k}$ if the soliton velocity is less than the minimum phase velocity

$$|\mathbf{V}| < \min(\omega_k/k). \tag{6}$$

Conversely, the difference will be negative for all $\mathbf{k}$ if the soliton velocity is greater than the maximum phase velocity

$$|\mathbf{V}| > \max(\omega_k/k). \tag{7}$$

We will show that a soliton solution is possible if the condition (6) (or (7)) is satisfied. Let us assume the opposite to be true — let the conditions (6) and (7) be violated, i.e., the equation (1) has a solution. For simplicity, we will assume that it is unique: $\mathbf{k} = \mathbf{k}_0$. Then, since $x\delta(x) = 0$, the homogeneous linear equation

$$(\omega_k - \mathbf{k}\cdot\mathbf{V})C_k = 0$$

possesses a nontrivial solution in the form of a monochromatic wave

$$C_k = A\delta(\mathbf{k} - \mathbf{k}_0).$$

In this case (5) can be written (by virtue of the Fredholm alternative)

$$c_k = A\delta(\mathbf{k} - \mathbf{k}_0) + \frac{f_k}{\omega_k - \mathbf{k}\cdot\mathbf{V}} \quad \text{with} \quad f_{k_0} = 0. \tag{8}$$

This equation, in contrast to (5), contains a free parameter — the complex amplitude A. It can be solved, for example, by iterations, taking $A\delta(\mathbf{k} - \mathbf{k}_0)$ as the zeroth term. It is important that because of the nonlinearity as a result of iterations one will obtain multiple harmonics with $\mathbf{k} = n\mathbf{k}_0$ where n is integer. The solution will consist of a collection of δ-functions. Correspondingly, in physical space the solution will be a periodic function of the coordinates, i.e., it will be nonlocalized. Hence follows the first selection rule for solitons: the difference $\omega_k - \mathbf{k}\cdot\mathbf{V}$ must be sign-definite, which is equivalent to the requirements (6) or (7). In other words, it means the absence of Cherenkov radiation.

In this entire scheme, however, there is an important exception. Having represented (5) in the form (8), we have in fact assumed that the singularity in the expression

$$\frac{f_k}{\omega_k - \mathbf{k}\cdot\mathbf{V}} \tag{9}$$

is nonremovable. This may not be the case — the singularity in the denominator in (9) could be cancelled with the numerator, i.e., it could be removable [10]. For example, this happens for the classic soliton of the KdV equation, for equations which are generalizations of the KdV equation [16] for a combination of the one-dimensional NLS and the MKdV equations [10, 17] integrated by the same Zakharov–Shabat operator [18] and so on. In all of these cases cancellation occurs as a result of the k dependence of the matrix elements. However, even in these cases, the selection rule for solitons remains the same after the resonance (1) is removed — the part remaining in the denominator must be sign-definite.

In what follows the singularities in (9) are assumed to be nonremovable in the forbidden region, and we study the behavior of the soliton solution as the soliton velocity approaches the critical value. For definiteness, it is assumed that the plane $\omega = \mathbf{k}\cdot\mathbf{V}$ is tangent to the dispersion surface $\omega = \omega_k$ from below, i.e., the criterion (6) holds. Let touching occur at the point $\mathbf{k} = \mathbf{k}_0$. Then, instead of (8), in the allowed region

$$c_k = \frac{f_k}{\omega_k - \mathbf{k}\cdot\mathbf{V}}.$$

As the velocity V approaches the critical value V_{cr}, the denominator in this expression becomes small near the touching point, so that c_k gets a sharp peak at this point

$$c_k = \left[\frac{1}{2}\omega_{\alpha\beta}\kappa_\alpha\kappa_\beta + k_0(V_{cr} - V)\right]^{-1} f_k. \tag{10}$$

Here $\omega_{\alpha\beta} = \partial^2\omega/\partial k_\alpha \partial k_\beta$ is a symmetric, positive-definite, tensor of the second derivatives, evaluated at $\mathbf{k} = \mathbf{k}_0$, and $\kappa = \mathbf{k} - \mathbf{k}_0$.

It is evident from (10) that as V approaches the critical velocity, the width of the peak narrows as $\sqrt{V_{cr} - V}$, and the distribution corresponding to the main peak $\mathbf{k} = \mathbf{k}_0$ approaches a monochromatic wave. Accounting for nonlinearity, the spectrum contains harmonics which are multiples of $\mathbf{k} = \mathbf{k}_0$. If it is assumed that the amplitude of the soliton vanishes gradually as $V \to V_{cr}$ (which would correspond to a second-order phase transition), then the solution $\psi(\mathbf{x})$ (or, equivalently, c_k) can be sought as an expansion in terms of harmonics:

$$\psi(\mathbf{x}') = \sum_{h=-\infty}^{\infty} \psi_n(\mathbf{X}) e^{ih\mathbf{k}_0\cdot\mathbf{x}'}, \quad \mathbf{x}' = \mathbf{x} - \mathbf{V}t. \tag{11}$$

Here the small parameter

$$\lambda = \sqrt{1 - V/V_{cr}} \tag{12}$$

and the "slow" coordinate $\mathbf{X} = \lambda \mathbf{x}'$ are formally introduced, so that $\psi_n(\mathbf{X})$ is the amplitude of the envelope of n-th harmonic. The assumption that the soliton amplitude vanishes continuously at $V = V_{cr}$ means that the leading term of the series in (11) corresponds to the first harmonic, and all other harmonics are small in the parameter λ. This is the condition under which the nonlinear Schrödinger equation is derived (see, for example, [10, 19, 2]). In the present case, we arrive at the stationary NLS

$$-k_0 V_{cr} \lambda^2 \psi_1 + \frac{1}{2}\omega_{\alpha\beta}\frac{\partial^2 \psi_1}{\partial X_\alpha \partial X_\beta} + B|\psi_1|^2\psi_1 = 0 \tag{13}$$

at leading order in λ, where B is related to the matrix $\tilde{T}_{k_1k_2k_3k_4}$ of four-wave interactions as

$$B = -(2\pi)^d \tilde{T}_{k_0k_0k_0k_0}. \tag{14}$$

In this approximation the leading term in the interaction Hamiltonian has the form

$$H_{int} = \frac{\tilde{T}_{k_0k_0k_0k_0}}{2}\int c^*_{k_1} c^*_{k_2} c_{k_3} c_{k_4} \delta_{\mathbf{k_1}+\mathbf{k_2}-\mathbf{k_3}-\mathbf{k_4}} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 = -\frac{B}{2}\int |\psi_1|^4 d\mathbf{x}, \tag{15}$$

and the tilde means renormalization of the vertex due to the three-wave interaction — in the present case the interaction with the zeroth and second harmonics. As we have already noted, $\omega_{\alpha\beta}$ in (13) is a symmetric positive-definite tensor. For this reason, performing a rotation to its principal axes and carrying out the corresponding extensions along each axis, (13) can be transformed into the standard form

$$-\lambda^2\psi + \Delta\psi - \mu|\psi|^2\psi = 0, \tag{16}$$

where $\mu = \text{sign}(\tilde{T}\omega_{\alpha\alpha})$. Hence it follows, in the first place, that solitons are possible only if μ is negative (focusing nonlinearity when the product $\tilde{T}\omega_{\alpha\alpha}$ is negative) and, in the second place, that the amplitude of the solitons is proportional to

$$\lambda = \sqrt{1 - V/V_{cr}},$$

i.e., the amplitude vanishes according to a square-root law, the size of the soliton increases as $1/\lambda$ as the velocity approaches the critical value.

In order to illustrate how this mechanism works consider the simplest example, i.e., the time-dependent one-dimensional nonlinear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} + \psi_{xx} + 2|\psi|^2\psi = 0. \tag{17}$$

As is well known, this equation, unlike the general equation (3), has one additional symmetry, namely the gradient symmetry $\psi \to \psi e^{i\phi}$. To find the corresponding solution one should put $\psi(x,t) = e^{i\nu t}\psi(x - Vt)$, where ψ obeys the equation

$$L(-i\partial_x)\psi \equiv iV\psi_x + \nu\psi - \psi_{xx} = 2|\psi|^2\psi. \tag{18}$$

For the present case, in accordance with (1), the condition for Cherenkov radiation will be written as follows:

$$kV = \Omega(k) \quad \text{or} \quad L(k) = 0\ , \tag{19}$$

where the dispersion law for the equation (18) takes the form

$$\Omega(k) = \nu + k^2. \tag{20}$$

Hence one can see that for $\nu < 0$ the resonance condition (19) is satisfied for any value of the velocity V! Consequently solitons do not exist in this case. This can be checked directly by solving equation (18): for $\nu < 0$ all solutions are periodic or quasi-periodic. Soliton solutions are possible for positive ν. Their velocities lie in the range $-2\sqrt{\nu} \le V \le 2\sqrt{\nu}$. At the points $k = \pm\sqrt{\nu}$ the dispersive curve $\Omega = \Omega(k)$ touches the straight line $\Omega = kV_{cr}$. At these points the solution must vanish in agreement with the general considerations. It directly follows from the exact solution of (18):

$$\psi = e^{i\nu t}\frac{e^{iVx'/2}\Delta k}{\cosh(\Delta k x')},\quad x' = x - Vt,\quad \Delta k = \sqrt{\nu - V^2/4}. \tag{21}$$

Solitons exist only for $\nu > V^2/4$. The upper boundary in this inequality defines the critical velocity

$$V_{cr} = 2\sqrt{\nu}.$$

It is important to note also that for $\nu > V^2/4$ the operator L in the equation (18) is positive definite.

3. Stability of Solitons

To include the time dependence in the averaged equations the amplitudes ψ_n in the expansion (11) must be assumed to depend not only on the "slow" coordinate X but also on the slow time $T = \lambda^2 t$. Then a multiscale expansion gives the nonstationary analog of the NLSE

$$i\psi_t - \lambda^2\psi + \Delta\psi - \mu|\psi|^2\psi = 0 \tag{22}$$

instead of the stationary NLSE (16). The soliton stability problem for this equation has been well studied (see, for example, [10] and [21]). We recall the basic points in the investigation of stability. The equation (22) as an equation for envelopes inherits the canonical Hamiltonian form (3)

$$i\frac{\partial\psi}{\partial t} = \frac{\delta\tilde{H}}{\delta\psi^*}, \tag{23}$$

where the Hamiltonian

$$\tilde{H} = \lambda^2 N + \int (|\nabla\psi|^2 - |\psi|^4)d\mathbf{r}, \qquad (\mu = -1), \tag{24}$$

arises as a result of averaging the initial Hamiltonian. The equation (22) preserves, besides $\tilde{H}$, the total number N of particles (adiabatic invariant), so that solitons are stationary points of the energy functional $E = \tilde{H} - \lambda^2 N$ with the fixed number of particles

$$\delta(E + \lambda^2 N) = 0. \tag{25}$$

The number of particles (or intensity) in a soliton solution as a function of λ has the form

$$N_s = \int |\psi|^2 d\mathbf{x} = \lambda^{2-d} \int |g(\xi)|^2 d\xi, \tag{26}$$

where d is the dimension of the space, and $g(\xi)$ satisfies the equation

$$-g + \Delta g + |g|^2 g = 0.$$

In the one-dimensional case $g = \sqrt{2}\,\mathrm{sech}\,\xi$ and, correspondingly, $N_s = 4\lambda$. In the two-dimensional case N_s is independent of λ for the entire family of solitons, while in the three-dimensional case N_s decreases with increasing λ. The dependence of N_s on λ^2 is crucial from the standpoint of soliton stability. It is obvious that the most dangerous disturbances will be those having wave numbers close to $\mathbf{k} = \mathbf{k}_0$ moving together with the soliton, i.e., modulation-type disturbances.

According to (25) the envelope solitons are stationary points of the energy E for a fixed number of waves N. Therefore such solutions will be stable in the Lyapunov sense if they realize a minimum (or a maximum) of the energy for fixed N.

Consider first the scaling transformations leaving N unchanged,

$$\psi(\mathbf{x}) = \frac{1}{a^{d/2}}\psi_s\left(\frac{\mathbf{x}}{a}\right), \tag{27}$$

where ψ_s is the solitonic solution. The energy E under this transformation becomes a function of the scaling parameter a:

$$E(a) = \frac{I_{1s}}{a^2} - \frac{I_{2s}}{a^d}, \tag{28}$$

where $I_{1s} = \int |\nabla\psi_s|^2 d\mathbf{x}$, $I_{2s} = \int |\psi_s|^4 d\mathbf{x}$ and $\mu = -1$. Hence it is easy to see that in the one-dimensional case the energy (28) is bounded from below and has a minimum at $a = 1$ corresponding to the soliton solution. In this case

$$E_s = -\frac{2\lambda^3}{3} \quad \text{and} \quad 2I_{1s} = I_{2s} = \frac{4\lambda^3}{3}.$$

The soliton also realizes a minimum of E with respect to another simple transformation, i.e., the gauge one, $\psi_0(x) \to \psi_0(x)\exp[i\chi(x)]$, which also preserves N,

$$E = E_s + \int (\chi_x)^2 \psi_0^2 dx.$$

Thus, for $d = 1$ both simple transformations yield a minimum for the energy, thus indicating soliton stability for the one-dimensional geometry.

Now we give an exact proof of this fact. The crucial point of this proof is based on integral estimations of Sobolev type. These inequalities arise as the sequences of general imbedding theorem. The (Sobolev) theorem says that a space L_p can be imbedded into the Sobolev space W_2^1 if the space dimension

$$D < \frac{2}{p}(p+4).$$

This means that between the norms

$$\|u\|_p = \left[\int |u|^p d^D x\right]^{1/p},\ p > 0,$$

$$\|u\|_{W_2^1} = \left[\int (\mu^2|u|^2 + |\nabla u|^2) d^D x\right]^{1/2},\ \mu^2 > 0,$$

there exists the following inequality (see, e.g., [23]):

$$\|u\|_p \le M\|u\|_{W_2^1}, \tag{29}$$

where M is some positive constant. For $D = 1$ and $p = 4$, the inequality (29) is

$$\int_{-\infty}^{\infty} |\psi|^4 dx \le M_1 \left[\int_{-\infty}^{\infty} (\mu^2|\psi|^2 + |\psi_x|^2) dx\right]^2. \tag{30}$$

Here it is straightforward to get a multiplicative variant of the Sobolev inequality, the so-called Gagliardo–Nirenberg inequality (GNI) [24] (see also [23, 25, 26]).

Use the scaling transform $x \to \alpha x$ in (29). Instead of (30) we have

$$\int_{-\infty}^{\infty} |\psi|^4 dx \le M_1 \left[\mu^2 \int_{-\infty}^{\infty} |\psi|^2 dx \cdot \alpha + \int_{-\infty}^{\infty} |\psi_x|^2 dx \cdot \frac{1}{\alpha}\right]^2.$$

This inequality holds for any (positive) α including a minimal value for the r.h.s. The computing of its minimum yields the GNI:

$$I_2 \leq C N^{3/2} I_1^{1/2}, \tag{31}$$

where $I_1 = \int |\psi_x|^2 dx$, $I_2 = \int |\psi|^4 dx$, and C is a new constant. Then this inequality can be improved by finding the best (minimal possible) constant C.

To find C_{best} consider all extrema of the functional

$$J\{\psi\} = \frac{I_2}{N^{3/2} I_1^{1/2}}. \tag{32}$$

The latter problem reduces to the solution of stationary NLSE (16):

$$-\lambda^2 \psi + \psi_{xx} + 2|\psi|^2 \psi = 0.$$

Hence we find that the best constant C_{best} is a value of $J\{\psi\}$ on the soliton solution:

$$C_{best} = \frac{I_{2s}}{N^{3/2} I_{1s}^{1/2}} = \frac{2 I_{1s}^{1/2}}{N^{3/2}}. \tag{33}$$

This inequality allows us to obtain immediately a proof of 1D soliton stability. Substituting (31) into (24) results the following estimation for the energy:

$$E \geq I_1 - C_{best} I_1^{1/2} N^{3/2} = E_s + (I_1^{1/2} - I_{1s}^{1/2})^2. \tag{34}$$

The inequality becomes precise on the soliton solution, thus proving its stability. Note that this provides the stability of solitons not only with respect to small perturbations, but also against finite perturbations.

In the three-dimensional case, in contrary, the function $E(a)$ in (28) has a maximum, corresponding to the soliton solution, and is unbounded from below as $a \to 0$. The gauge transformation gives a minimum of E and therefore all soliton solutions at $d = 3$ represent saddle points of the energy. That indicates a possible instability of solitons in this case.

The fact of (linear) instability of three-dimensional solitons follows from the Vakhitov-Kolokolov criterion [20]. It is as follows: *if*

$$\frac{\partial N_s}{\partial \lambda^2} > 0, \tag{35}$$

then the soliton is stable and, respectively, unstable if this derivative is negative.

This criterion has a simple physical meaning. The value $-\lambda^2$ for the stationary nonlinear Schrödinger equation (16) can be interpreted as the energy of the bound state–soliton. If we add one "particle" to the system and the energy of this bound state decreases, then we have a stable situation. If by adding one "particle" the level $-\lambda^2$ is pushed towards the continuous spectrum, then such a soliton is unstable.

At $d = 3$ the derivative $\partial N_s/\partial\lambda^2 < 0$ and therefore 3D solitons are unstable (the modulational instability). For the two-dimensional case the Vakhitov–Kolokolov criterion (35) gives an absence of linear exponential instability. A more detailed analysis in this case yields the power type instability (for details see the survey [21] and [22]).

Thus, the solitons are stable only in the one-dimensional case, while in the two-dimensional (critical) and three-dimensional cases they are unstable and can be considered as separatrix solutions separating collapsing solutions from the dispersive ones [27].

This is probably the simplest method for explaining the well-known empirical fact that solitons, as a rule, exist only in one-dimensional systems. For multidimensional systems the stable solitons are rare and can only appear as a result of topological constraints or of a mechanism that removes Cherenkov singularities (which is discussed in the present paper). The latter, as can be easily understood, is due to the existence of a certain class of symmetry.

4. From Supercritical to Subcritical Bifurcations

For subcritical bifurcation at the critical velocity the soliton undergoes a jump in its amplitude. In this case the corresponding theory can be developed near the transition point between subcritical and supercritical bifurcations (in analogy with the tri-critical point for phase transitions). In the series of papers [11, 13, 28–30] it was shown that in this case the soliton behavior can be described by means of the generalized nonlinear Schrödinger equation (NLSE) for the envelope ψ, which in the one-dimensional case is as follows:

$$i\frac{\partial\psi}{\partial t} - \lambda^2\omega_0\psi + \frac{\omega_0''}{2}\psi_{xx} - \mu|\psi|^2\psi + 4i\beta|\psi|^2\psi_x + \gamma\psi\widehat{k}|\psi|^2 + 3C|\psi|^4\psi = 0, \quad (36)$$

where $\omega_0 \equiv \omega(k_0)$ and k_0 are the carrying frequency and the wave number, respectively, $\lambda^2 = (V_{cr} - V)/V_{cr} \ll 1$, ω_0'' the second derivative of $\omega(k)$ taken at $k = k_0$. Here the four-wave coupling coefficient μ is assumed to have additional smallness characterizing the proximity to the transition from supercritical to subcritical bifurcations. The transition point is defined from the equation $\mu = 0$. For example, for the interfacial deep-water waves propagating along the interface between two ideal fluids in the presence of capillarity [28, 29]

$$\mu = \frac{k_0^3}{1+\rho}\left(A_{cr}^2 - A^2\right),$$

where ρ is the density ratio, $A = (1-\rho)/(1+\rho)$ the Atwood number, $A_{cr}^2 = 5/16$ and $\rho_{cr} = (21 - 8\sqrt{5})/11$, as it was shown in [13]. For $\rho < \rho_{cr}$, the four-wave coupling coefficient μ is negative, and the corresponding nonlinearity is of the focusing type. In this case, the solitary waves near the critical velocity V_{cr} are described by the stationary ($\partial/\partial t = 0$) NLSE and undergo a supercritical bifurcation at $V = V_{cr}$ [13]. For $\rho > \rho_{cr}$, the coupling coefficient changes sign and, as a result, the bifurcation becomes subcritical. For water waves in finite depth h the coefficient μ changes its sign at $\theta_{cr} = k_0 h \approx 1.363$ [31] while ω_0'' is always negative. Thus the nonlinearity belongs to the focusing type for $\theta(= kh) > \theta_{cr}$ and respectively becomes defocusing in the region $\theta < \theta_{cr}$ [31, 32]. In nonlinear optics, as shown in [11], a decrease of μ (Kerr constant) can be provided by the interaction of light pulses with acoustic waves (Mandelstamm–Brillouin scattering).

Because of smallness of μ we keep in (36) the following order nonlinear terms: the gradient term ($\sim \beta$) responsible for self-steepening of the pulse (analog of the Lifshitz term in phase transitions), the nonlocal term (due to presence of the integral operator $\widehat{k}$, the Fourier transform of its kernel is equal to $|k|$) and the six-wave nonlinear term with coupling coefficient C. Two additional 4-wave interaction terms, both local and nonlocal, appear as a result of expansion of the four-wave matrix element $T_{k_1k_2k_3k_4}$ in powers of the small parameters $\kappa_i = k_i - k_0$:

$$T_{k_1k_2k_3k_4} = \frac{\mu}{2\pi} + \frac{\beta}{2\pi}(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4) \tag{37}$$
$$-\frac{\gamma}{8\pi}(|\kappa_1 - \kappa_3| + |\kappa_2 - \kappa_3| + |\kappa_2 - \kappa_4| + |\kappa_1 - \kappa_4|).$$

The existence of nonlocal contribution in the expansion is connected with non-analytical dependence of the matrix element T in its arguments. For interfacial deep-water waves (IW) this nonanalyticity originates from the solution of Laplace equation for the hydrodynamic potential and its reduction to the moving interface. For instance, for water waves (WW) with a finite depth the nonlocal term is absent [32] as well as for electromagnetic waves in nonlinear dielectrics [11] because of the analyticity of matrix elements with respect to frequencies, which is a consequence of causality (see, for example, Refs. [11, 33]). In the latter case the spatial dispersion effects are relativistically small and can be neglected.

For both IW and WW near the transition point, $\omega_0'' C$ is positive; moreover, γ is also positive for IW, and therefore the corresponding nonlinearities are focusing, thus providing the existence of localized solutions. Depending on the sign of μ there exist two branches of solitons. For IW they were found numerically [28, 29] using the Petviashvili scheme [34]. Explicit solutions for both kinds of IW solitons can be obtained in the limiting case only when $V \to V_{cr}$. For negative μ these are the classical NLS solitons with a sech shape. For the subcritical bifurcation at $V = V_{cr}$ the soliton amplitude remains finite with algebraic decay ($\sim 1/|x|$)

at infinity [28, 29]. When a nonlocal nonlinearity is absent ($\gamma = 0$), the soliton solutions can be found explicitly. For both branches at large λ the number of waves $N = \int |\psi|^2 dx$ approaches from below and above the same value N_{cr} which coincides with the number of waves N on the solitons with $\mu = 0$. For the solitons in fibers this property means that the energy of optical pulse saturates tending to the constant value with a decrease of the pulse duration.

On the other hand, all solitons of (36) are stationary points of the energy E for fixed number of waves : $\delta(E + \lambda^2 N) = 0$, where the energy in dimensionless variables is given by

$$E = \int \left[|\psi_x|^2 + \frac{\mu}{2}|\psi|^4 + i\beta(\psi_x^*\psi - \psi_x\psi^*)|\psi|^2 - \frac{\gamma}{2}|\psi|^2\hat{k}|\psi|^2 - C|\psi|^6 \right] dx. \quad (38)$$

This allows one to use the Lyapunov theorem in the analysis of their stability. Here, for the IW, $\lambda \sim (V_{cr} - V)^{1/2}|\rho - \rho_{cr}|^{-1}$, $C = 319/1281$ and

$$\mu = \mathrm{sign}(\rho - \rho_{cr}), \quad \beta = 6/\sqrt{427}, \quad \gamma = 32/\sqrt{427}, \quad (39)$$

and for the WW case (compare with [32])

$$\mu = \mathrm{sign}(\theta_{cr} - \theta), \ \ \beta \approx -0.397, \ \ C \approx 0.176. \quad (40)$$

As shown in [11, 28, 29, 32], for $N < N_{cr}$ solitons corresponding to the supercritical branch realize the minimum values of the energy and therefore they are stable in the Lyapunov sense, i.e., stable with respect to not only small perturbations but also against finite ones. In particular, the boundedness of E from below can be viewed if one considers the scaling transformation $\psi = (1/a)^{1/2}\psi_s(x/a)$ retaining the number of waves N, where $\psi = \psi_s(x)$ is the soliton solution. Under this transform E becomes a function of the scaling parameter a:

$$E\,(a) = \left(\frac{1}{a} - \frac{1}{2a^2}\right)\frac{\mu}{2}\int |\psi_s|^4\, dx. \quad (41)$$

It is worth noting that the dispersion term and all nonlinear terms in E, except $\int \frac{\mu}{2}|\psi|^4 dx$, have the same scaling dependence $\propto a^{-2}$. The latter means that at $\mu = 0$ (36) can be related to the critical NLS equation like the two-dimensional cubic NLS equation. From (41) it is also seen that for $\mu < 0$ $E(a)$ has a minimum corresponding to the soliton. Unlike in the supercritical case, the scaling transformation for another soliton branch with $\mu > 0$ gives a maximum of $E(a)$ on solitons and unboundedness of E as $a \to 0$. Under the gauge transformation $\psi = \psi_s e^{i\chi}$, on the contrary, the energy reaches a minimum on soliton solutions and consequently the solitons with $\mu > 0$ represent saddle points. This indicates a possible instability of solitons for the whole subcritical branch, at least with respect to finite perturbations.

We consider this question in detail and emphasize a nonlinear stage of instability following to our recent paper [30]. This problem, indeed, is not trivial in spite of a close similarity with the critical NLSE. It is worth noting that (36) at $\mu = \gamma = C = 0$ represents an integrable model (the so-called derivative NLSE) [35], and exponentially decaying solitons in this model are stable. It is more or less evident also that small coefficients γ, C cannot break the stability of solitons. This means that in the space of parameters we may expect the existence of a threshold. Above this, the threshold solitons must be unstable and the development of this instability may lead to collapse, i.e., the formation of a singularity in finite time.

Consider the energy (38) written in terms of amplitude r and phase φ ($\psi = re^{i\varphi}$):

$$E = \int \left[r_x^2 + \frac{\mu}{2} r^4 - \frac{\gamma}{2} r^2 \widehat{k} r^2 - \frac{1}{3} r^6 + r^2 \left(\varphi_x + \beta r^2 \right)^2 \right] dx, \tag{42}$$

where by an appropriate choice of the new dimensionless variables the renormalized constant $\tilde{C} = C + \beta^2$ can be taken equal to $1/3$. Hence one can see that the energy takes its minimum value when the last term in (42) vanishes, i.e. when

$$\varphi_x + \beta r^2 = 0. \tag{43}$$

The integration of this equation gives an $x-$dependence for the phase that is called the chirp in nonlinear optics. It is interesting to note that the remaining part of the energy does not contain the phase at all.

First, study the local model when $\gamma = 0$. Let the energy be negative in some region Ω : $E_\Omega < 0$. Then, following [36, 26], one can establish that due to radiation of small amplitude waves $E_\Omega < 0$ can only decrease, becoming more and more negative, but the maximum value of $|\psi|$, according to the mean value theorem, can only increase:

$$\max_{x \in \Omega} |\psi|^4 \geq \frac{3|E_\Omega|}{N_\Omega}. \tag{44}$$

This process is possible only for the energies which are unbounded from below. In accordance with (41) such a situation is realized when $\mu > 0$. In this case the radiation leads to the appearance of infinitely large amplitudes r. However, it is impossible to conclude that the singularity formation develops in finite time.

For $\gamma > 0$ the estimations on the maximum value of $|\psi|$ are not as transparent as they are for the local case. Instead of (44), it is possible to obtain a similar estimate

$$\max_{x} |\psi|^4 \geq \frac{3|E|}{N}.$$

However, it is expressed through the total energy E and the total number of waves N. Besides, two inequalities must be satisfied: $E < 0$ and $N < \frac{2N_2}{\gamma}$.

For interfacial waves, $N_2 \approx 1.39035 > N_{cr} \approx 1.3521$. Thus, the maximum amplitude in this case is bounded from below by a conservative quantity and this maximum can never disappear during the nonlinear evolution.

Now we consider the situation where the self-steepening process can be neglected ($\beta = 0$). In this case (36) becomes

$$i\psi_t + \psi_{xx} - \lambda^2\psi - \mu|\psi|^2\psi + \gamma\psi\widehat{k}|\psi|^2 + 3C|\psi|^4\psi = 0.$$

It is possible to obtain a criterion of collapse using the virial equation (for details, see [37, 36, 38]). This equation is written for the positive definite quantity

$$R = \int x^2|\psi|^2 dx,$$

which, up to the multiplier N, coincides with the mean square size of the distribution. The second derivative of R with respect to time is defined by the virial equation

$$R_{tt} = 8\left(E - \frac{\mu}{4}\int|\psi|^4 dx\right). \tag{45}$$

Hence, for $\mu > 0$ one can easily obtain the following inequality:

$$R_{tt} < 8E,$$

which yields, after double integration, $R < 4Et^2 + \alpha_1 t + \alpha_2$. Here $\alpha_{1,2}$ are constants which are obtained from the initial conditions. Hence, it follows that for the states with negative energy, $E < 0$, there always exists such a moment of time t_0 when the positive definite quantity R vanishes. At this moment of time the amplitude becomes infinite. Therefore the condition $E < 0$ represents a sufficient criterion of collapse (compare with [37, 36]). However, it is necessary to add that this criterion can be improved in the same way as in [39, 27] for the three-dimensional cubic NLS equation. From (45) one can see that for the stationary case (on the soliton solution) $E_s = \frac{\mu}{4}\int|\psi_s|^4 dx$, in agreement with (41). As it was shown before, for $\mu > 0$ the soliton realizes a saddle point of E for fixed N. It follows from (41) that for small a the energy becomes unbounded from below, but for $a > 1$ it decreases (this corresponds to spreading). Therefore in order to achieve a blow-up regime the system should pass through the energetic barrier equal to E_s. Thus, for this case the criterion $E < 0$ must be changed into the sharper criterion: $E < E_s$. This criterion can be obtained rigorously using step by step the scheme presented in [39, 27] and therefore we skip its derivation.

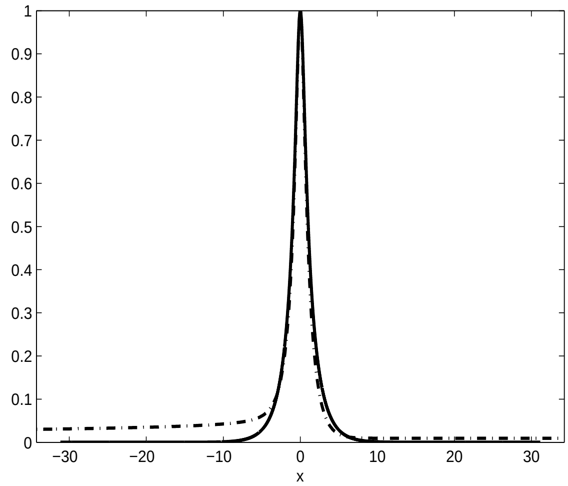

Fig. 1: Initial (solid line) and final (dashed line) at $t = 1.18$ distributions for $|\overline{\psi}|$, interfacial waves, selfsimilar variables. The soliton amplitude was increased by 1%, $\mu = 1$, $\lambda = 1$. The ratio between final and initial soliton amplitudes in the physical variables is about 11.

In order to verify all the theoretical arguments about the formation of collapse presented above we performed a numerical integration of the NLSE (36) for $\mu > 0$ by using the standard 4th order Runge–Kutta scheme. The initial conditions were chosen in the form of solitons but with larger amplitudes than for the stationary solitons. Increasing in the initial amplitude varied in the interval from 0.1% up to 10%. The initial phase was given by means of (43). In all runs with theses initial conditions we observed a high increase of the soliton amplitude up to a factor 14 with a shrinking of its width. In a peak region the pulses for both IW and WW cases behaved similarly. Near the maximum the pulse peak was almost symmetric: anisotropy was not visible. The difference was observed in the asymptotic regions far from the pulse core, where the pulses had different asymmetries for IW and WW because of the opposite sign for β (see (39), (40)). For the given values of β we did not observe the simultaneous formation of two types of singularities with blowing-up amplitudes and sharp gradients as it was demonstrated in the recent numerical experiments for the three-dimensional collapse of short optical pulses due to self-focusing and self-steepening in the framework of the generalized NLS equation [40] and equations of the Kadomtsev–Petvishvili type [41].

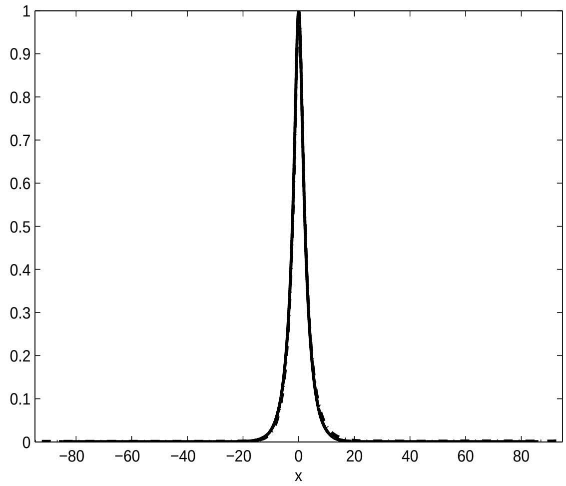

Fig. 2: Initial (solid line) and final (dashed line) at $t = 2.7192$ distributions for $|\overline{\psi}|$, WW solitons, selfsimilar variables. The soliton amplitude was increased by 1%, $\mu = 1$, $\lambda = 1$. The ratio between final and initial soliton amplitudes in the physical variables is about 11.

In our numerical computations we found that the amplitude and its spatial collapsing distribution developed in a selfsimilar manner. Near the collapse point in the equation (with $\mu > 0$) one can neglect the term proportional to μ. In this asymptotic regime (36) admits selfsimilar solutions,

$$r(x,t) = (t_0 - t)^{-1/4} f\left(\frac{x}{(t_0 - t)^{1/2}}\right), \tag{46}$$

where t_0 is the collapse time.

To verify that we approached the asymptotic behavior given by (46), at each moment of time we normalized the ψ-function by the maximum (in x) of its modulus $\max|\psi| \equiv M$ and introduced new self-similar variables,

$$\psi(x,t) = M\overline{\psi}(\xi,\tau),\ \xi = M^2(x - x_{max}),\ \ \tau = \ln M. \tag{47}$$

Here x_{max} is the point corresponding to the maximum of $|\psi|$. In comparison with those given by (46), new variables are more convenient because they do not require the determination of the collapsing time t_0.

Fig. 1 and Fig. 2 show typical dependences of $|\overline{\psi}|$ as a function of the self-similar variable ξ at $t = 0$ (solid line) and at the final time (dashed line) for

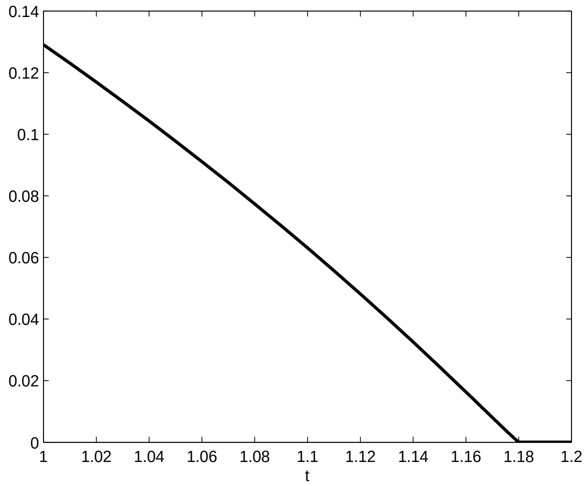

Fig. 3: Dependence of $1/\max|\psi|^4$ on time. Interfacial waves.

both the IW and WW cases. In both figures one can see a fairly good coincidence between the initial soliton distribution and the final one at the central (collapsing) part of the pulse and asymmetry of the pulse at its tails due to self-steepening. The latter demonstrates that the collapse has a selfsimilar behavior. The form of the central part of the pulse approaches the soliton shape because asymptotically the NLS model (36) tends to the critical NLS system. It should be mentioned that this has been well known for the classical two-dimensional NLS equation since the paper by Fraiman [42].

Fig. 3 shows how $1/\max|\psi|^4$ depends on time. This dependence is almost linear in the correspondence with the selfsimilar law (46). If the initial amplitudes were less than the stationary soliton values, then the distribution would spread in time dispersively, in full correspondence with qualitative arguments based on the scaling transformations (41).

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