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Functional Model of Commutative Operator Systems

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A functional model for a commutative system of the linear bounded operators $\{T_1, T_2\}$, when T_1 is a contraction, is built. The construction of functional model is based on an analogue with many parameters of the Lax – Phillips scattering scheme for the isometric dilation $U(n)$ of the semigroup with two parameters $T(n) = T_1^{n_1} T_2^{n_2}$, where $n = (n_1, n_2) \in \mathbb{Z}_+^2$.

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As it is well known, one of the most natural ways of constructing the functional model of contraction operator T ($\|T\| < 1$) is based on the Lax–Phillips scattering scheme [1]. In this work, the functional model of commutative system of the linear bounded operators $\{T_1, T_2\}$, $[T_1, T_2] = 0$, when T_1 is a contraction, is obtained using isometric extensions and an analogue with many variables of the Lax–Phillips scattering scheme [2–5].

It is shown that the weight matrices functions of model space have the form which is different from a traditional (the B.S. Pavlov model [1]) one and the structure of given weight functions itself is defined by external parameters of isometric extensions [2] of the operator system $\{T_1, T_2\}$. The functional model lies in the following: the operator T_1 is realized by means of operator of multiplication by independent variable in a special function space, the second operator T_2 represents the operator of multiplication by meromorphic operator function in the same space. It is typical of the constructed model to differ crucially from the well-known models in the nonselfadjoint case [6, 7].

1. Isometric Dilations of Commutative Operator System

I. Let a commutative system of the linear bounded operators $\{T_1, T_2\}$, $[T_1, T_2] = T_1 T_2 - T_2 T_1 = 0$, T_1 is a contraction, $\|T_1\| \le 1$, be given in the separable Hilbert space H. Following [2, 3, 8], define the commutative unitary extension for the system $\{T_1, T_2\}$.

Definition 1. *Let E and $\tilde{E}$ be the Hilbert spaces. The collection of mappings*

$$\begin{aligned} V_1 &= \begin{bmatrix} T_1 & \Phi \\ \Psi & K \end{bmatrix}; \quad V_2 = \begin{bmatrix} T_2 & \Phi N \\ \Psi & K \end{bmatrix} : \quad H \oplus E \to H \oplus \tilde{E}; \\ \overset{+}{V}_1 &= \begin{bmatrix} T_1^* & \Psi^* \\ \Phi^* & K^* \end{bmatrix}; \quad \overset{+}{V}_2 = \begin{bmatrix} T_2^* & \Psi^* \tilde{N}^* \\ \Phi^* & K^* \end{bmatrix} : \quad H \oplus \tilde{E} \to H \oplus E \end{aligned} \tag{1.1}$$

is said to be a commutative unitary extension of the commutative operator system T_1, T_2 in H, $[T_1, T_2] = 0$ if there are such operators σ, τ, N, Γ and $\tilde{\sigma}$, $\tilde{\tau}$, $\tilde{N}$, $\tilde{\Gamma}$ in the Hilbert spaces E and $\tilde{E}$, respectively, where σ, τ, $\tilde{\sigma}$, $\tilde{\tau}$ are selfadjoint, and the relations:

$$\begin{aligned} &1) \quad \overset{+}{V}_1 V_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}; \qquad V_1 \overset{+}{V}_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}; \\ &2) \quad V_2^* \begin{bmatrix} I & 0 \\ 0 & \tilde{\sigma} \end{bmatrix} V_2 = \begin{bmatrix} I & 0 \\ 0 & \sigma \end{bmatrix}; \quad \overset{+}{V}{}_2^* \begin{bmatrix} I & 0 \\ 0 & \tau \end{bmatrix} \overset{+}{V}_2 = \begin{bmatrix} I & 0 \\ 0 & \tilde{\tau} \end{bmatrix}; \\ &3) \quad T_2 \Phi - T_1 \Phi N = \Phi \Gamma; \qquad \Psi T_2 - \tilde{N} \Psi T_1 = \tilde{\Gamma} \Psi; \\ &4) \quad \tilde{N} \Psi \Phi - \Psi \Phi N = K \Gamma - \tilde{\Gamma} K; \\ &5) \quad \tilde{N} K = K N \end{aligned} \tag{1.2}$$

hold.

Consider the following class of commutative systems of the linear operators $\{T_1, T_2\}$ [3].

Definition 2. *The commutative operator system $\{T_1, T_2\}$ belongs to the class $C(T_1)$ and is said to be the contracting T_1 operator system if:*

$$\begin{aligned} &1)\ T_1 \textit{ is a contraction, } \|T_1\| \le 1; \\ &2)\ E = \overline{\tilde{D}_1 H} \supseteq \overline{\tilde{D}_2 H}; \quad \tilde{E} = \overline{D_1 H} \supseteq \overline{D_2 H}; \\ &3)\ \dim \overline{T_2 \tilde{D}_1 H} = \dim E; \quad \dim \overline{D_1 T_2 H} = \dim \tilde{E}; \\ &4)\ \textit{the operators } D_1|_{\tilde{E}}, \quad \tilde{D}_1 T_2^* \Big|_{\overline{T_2 \tilde{D}_1 H}}, \quad \tilde{D}_1 \Big|_E, \quad T_2^* D_1|_{\overline{D_1 T_2 H}} \\ &\textit{are boundedly invertible, where } D_s = T_s^* T_s - I,\ \tilde{D}_s = T_s T_s^* - I,\ s = 1, 2. \end{aligned} \tag{1.3}$$

It is easy to show that if $\{T_1, T_2\} \in C(T_1)$, then the unitary extension (1) always exists [2, 3].

II. Recall [1, 9, 10] the construction of unitary dilation U for the contraction T_1. Let $\mathcal{H}$ be the Hilbert space

$$\mathcal{H} = D_- \oplus H \oplus D_+, \tag{1.4}$$

where $D_- = l^2_{\mathbb{Z}_-}(E)$ and $D_+ = l^2_{\mathbb{Z}_+}(\tilde{E})$. Define the dilation U on the vector functions $f = (u_k, h, v_k)$ from $\mathcal{H}$ (1.4) in the following way:

$$Uf = \left(u_{k-1}, \tilde{h}, \tilde{v}_k\right), \tag{1.5}$$

where $\tilde{h} = T_1 h + \Phi u_{-1}$, $\tilde{v}_0 = \Psi h + K u_{-1}$, $\tilde{v}_k = v_{k-1}$, $k = 1, 2, \ldots$. The unitary property of U (1.5) in $\mathcal{H}$ follows from 1) (1.2).

The construction of isometric dilation [3] of the commutative operator system $\{T_1, T_2\} \in C\,(T_1)$ consists in the continuation of incoming D_- and outgoing D_+ subspaces

$$D_- = l^2_{\mathbb{Z}_-}(E); \quad D_+ = l^2_{\mathbb{Z}_+}(\tilde{E}) \tag{1.6}$$

by the second variable "n_2". Continue the functions u_{n_1} of $l^2_{\mathbb{Z}_-}(E)$ from semiaxis $\mathbb{Z}_-$ into domain

$$\tilde{\mathbb{Z}}^2_- = \mathbb{Z}_- \times (\mathbb{Z}_- \cup \{0\}) = \left\{n = (n_1, n_2) \in \mathbb{Z}^2 : n_1 < 0, n_2 \leq 0\right\} \tag{1.7}$$

using the Cauchy problem [2, 3].

$$\begin{cases} \tilde{\partial}_2 u_n = \left(N\tilde{\partial}_1 + \Gamma\right) u_n; \quad n = (n_1, n_2) \in \tilde{\mathbb{Z}}^2_-; \\ u_n|_{n_2=0} = u_{n_1} \in l^2_{\mathbb{Z}_-}(E) \end{cases} \tag{1.8}$$

where $\tilde{\partial}_1 u_n = u_{(n_1-1,n_2)}$, $\tilde{\partial}_2 u_n = u_{(n_1,n_2-1)}$. As a result, we obtain the Hilbert space $D_-(N,\Gamma)$ formed by the solutions u_n (1.8), besides, the norm in $D_-(N,\Gamma)$ is induced by the norm of initial data $\|u_n\| = \|u_{n_1}\|_{l^2_{\mathbb{Z}_-}(E)}$.

Similarly, continue the functions $v_{n_1} \in l^2_{\mathbb{Z}_+}(\tilde{E})$ from semiaxis $\mathbb{Z}_+$ into domain $\mathbb{Z}^2_+ = \mathbb{Z}_+ \times \mathbb{Z}_+$ using the Cauchy problem

$$\begin{cases} \tilde{\partial}_2 v_n = \left(\tilde{N}\tilde{\partial}_1 + \tilde{\Gamma}\right) v_n; \quad n = (n_1, n_2) \in \mathbb{Z}^2_+; \\ v_n|_{n_2=0} = v_{n_1} \in l^2_{\mathbb{Z}_+}(E). \end{cases} \tag{1.9}$$

Denote by $D_+(\tilde{N}, \tilde{\Gamma})$ the Hilbert space formed by solutions v_n (1.9), besides, $\|v_n\| = \|v_{n_1}\|_{l^2_{\mathbb{Z}_+}(\tilde{E})}$. Unlike the explicit recurrent scheme (1.8) of the layer-to-layer calculation of $n_2 \to n_2 - 1$ for u_n, in this case of constructing v_n in $\mathbb{Z}^2_+$, we have an implicit linear equation system for the layer-to-layer calculation of $n_2 \to n_2 + 1$ of function v_n.

Hereinafter, the following lemma [3] plays an important role.

Lemma 1.1. *Suppose the commutative unitary extension* $(V_s, \overset{+}{V}_s)$ *(1.1) is such that*

$$\operatorname{Ker}\Phi = \operatorname{Ker}\Psi^* = \{0\}. \tag{1.10}$$

Then $\operatorname{Ker} N \cap \operatorname{Ker} \Gamma = \{0\}$ *given* $\operatorname{Ker} K^* = \{0\}$, *and respectively* $\operatorname{Ker} \tilde{N}^* \cap \operatorname{Ker} \tilde{\Gamma}^* = 0$ *given* $\operatorname{Ker} K = \{0\}$.

The solvability of the Cauchy problem (1.9) follows from the given lemma [3].

Consider an operator function of the discrete argument

$$\tilde{\sigma}_\Delta = \begin{cases} I: & \Delta = (1,0); \\ \tilde{\sigma}: & \Delta = (0,1). \end{cases} \tag{1.11}$$

And let L_0^n be the nondecreasing broken line in $\mathbb{Z}_+^2$ that connects points $O = (0,0)$ and $n = (n_1, n_2) \in \mathbb{Z}_+^2$, the linear segments of which are parallel to the axes OX, $n_2 = 0$, and OY, $n_1 = 0$. By $\{P_k\}_0^N$ denote all integer-valued points from $\mathbb{Z}_+^2$, $P_k \in \mathbb{Z}_+^2$ $(N = n_1 + n_2)$ that lie on L_0^n, beginning with $(0,0)$ and finishing with point (n_1, n_2), that are numbered in nonascending order (of one of the coordinates P_k). Define the quadratic form

$$\langle \tilde{\sigma} v_k \rangle_{L_0^n}^2 = \sum_{k=0}^{N} \left\langle \tilde{\sigma}_{P_k - P_{k-1}} v_{P_k}, v_{P_k} \right\rangle \tag{1.12}$$

on the vector functions $v_k \in D_+(\tilde{N}, \tilde{\Gamma})$ assuming that $P_{-1} = (-1, 0)$.

Similarly, consider the nonincreasing broken line L_m^{-1} in $\tilde{\mathbb{Z}}_-^2$ (1.7) that connects points $m = (m_1, m_2) \in \tilde{\mathbb{Z}}_-^2$ and $(-1, 0)$, the linear segments of which are parallel to OX and OY. And let $\{Q_s\}_M^{-1}$, $M = m_1 + m_2$, be the integer-valued points on L_m^{-1}, beginning with $m = (m_1, m_2)$ and finishing with $(-1, 0)$, that are numbered in nondescending order (of one of the coordinates Q_s). In $D_-(N, \Gamma)$ define the metric

$$\langle \sigma u_k \rangle_{L_m^{-1}}^2 = \sum_{s=M}^{-1} \left\langle \sigma_{Q_s - Q_{s-1}} u_{Q_s}, u_{Q_s} \right\rangle, \tag{1.13}$$

besides $Q_M - Q_{M-1} = (1, 0)$ and the operator function σ_Δ is defined similarly to $\tilde{\sigma}_\Delta$ (1.11). Denote by $\tilde{L}_{-n}^{-1}$ the broken line in $\tilde{\mathbb{Z}}_-^2$ that is obtained from the curve L_0^n in $\mathbb{Z}_+^2$, $n \in \mathbb{Z}_+^2$, using the shift by "n" —

$$\tilde{L}_{-n}^{-1} = \left\{ Q_s = (l_1, l_2) \in \tilde{\mathbb{Z}}_-^2 : (l_1 + n_1 + 1, l_2 + n_2) = P_k \in L_0^n \right\}. \tag{1.14}$$

III. Having the Hilbert space $D_-(N, \Gamma)$, that is formed by the solutions of the Cauchy problem (1.8), and the space $D_+(\tilde{N}, \tilde{\Gamma})$, that is formed by the solutions of (1.9), define the Hilbert space

$$\mathcal{H}_{N,\Gamma} = D_-(N, \Gamma) \oplus H \oplus D_+(\tilde{N}, \tilde{\Gamma}), \tag{1.15}$$

in which the norm is defined by the norm of the initial space $\mathcal{H} = D_- \oplus H \oplus D_+$ (1.4). Denote by $\hat{\mathbb{Z}}_+^2$ the subset in $\mathbb{Z}_+^2$,

$$\hat{\mathbb{Z}}_+^2 = \mathbb{Z}_+^2 \backslash (\{0\} \times \mathbb{N}) = \{(0,0)\} \cup (\mathbb{N} \times \mathbb{Z}_+), \tag{1.16}$$

that obviously is an additional semigroup.

For every $n \in \hat{\mathbb{Z}}^2_+$ (1.16), define the operator function $U(n)$ that acts on the vectors $f = (u_k, h, v_k) \in \mathcal{H}_{N,\Gamma}$ (1.15) in the following way:

$$U(n)f = f(n) = (u_k(n), h(n), v_k(n)), \tag{1.17}$$

where $u_k(n) = P_{D_-(N,\Gamma)} u_{k-n}$ ($P_{D_-(N,\Gamma)}$ is an orthoprojector that corresponds with the restriction on $D_-(N,\Gamma)$); $h(n) = y_0$, besides $y_k \in H$, $k \in \mathbb{Z}^2_+$, is a solution of the Cauchy problem

$$\begin{cases} \tilde{\partial}_1 y_k = T_1 y_k + \Phi u_{\tilde{k}}; \\ \tilde{\partial}_2 y_k = T_2 y_k + \Phi N u_{\tilde{k}}; \\ y_n = h; \quad k = (k_1, k_2) \in \mathbb{Z}^2_+, \quad 0 \le k_1 \le n_1 - 1, \quad 0 \le k_2 \le n_2; \end{cases} \tag{1.18}$$

at the same time $\tilde{k} = k - n$ when $0 \le k_1 \le n_1 - 1$, $0 \le k_2 \le n_2$, and, finally,

$$v_k(n) = \hat{v}_k + v_{k-n} \tag{1.19}$$

and $\hat{v}_k = K u_{\tilde{k}} + \Psi y_k$, where y_k is a solution of the Cauchy problem (1.18).

In [3] it is shown that the operator function $U(n)$ (1.17) has the semigroup property and is the isometric dilation of the semigroup

$$T(n) = T_1^{n_1} T_2^{n_2}, \quad n = (n_1, n_2) \in \mathbb{Z}^2_+. \tag{1.20}$$

IV. Make a similar continuation of subspaces D_- and D_+ (1.6) from semiaxes $\mathbb{Z}_-$ and $\mathbb{Z}_+$ by the second variable "n_2", corresponding to the dual situation. By $D_+\left(\tilde{N}^*, \tilde{\Gamma}^*\right)$ denote the Hilbert space generated by the solutions $\tilde{v}_n$ of the Cauchy problem

$$\begin{cases} \partial_2 \tilde{v}_n = \left(\tilde{N}^* \partial_1 + \tilde{\Gamma}^*\right) \tilde{v}_n; \quad n = (n_1, n_2) \in \mathbb{Z}^2_+; \\ \tilde{v}_n|_{n_2=0} = v_{n_1} \in l^2_{\mathbb{Z}_+}(\tilde{E}), \end{cases} \tag{1.21}$$

in which the norm is induced by the norm of initial data $\|\tilde{v}_n\| = \|v_{n_1}\|_{l^2_{\mathbb{Z}_+}(E)}$, besides $\partial_1 \tilde{v}_n = \tilde{v}_{(n_1+1, n_2)}$, $\partial_2 \tilde{v}_n = \tilde{v}_{(n_1, n_2+1)}$.

Continue now every function $u_{n_1} \in l^2_{\mathbb{Z}_-}(E)$ into domain $\tilde{\mathbb{Z}}^2_-$ (1.7) using the Cauchy problem

$$\begin{cases} \partial_2 \tilde{u}_n = (N^* \partial_1 + \Gamma^*) \tilde{u}_n; \quad n = (n_1, n_2) \in \tilde{\mathbb{Z}}^2_-; \\ \tilde{u}_n|_{n_2=0} = u_{n_1} \in l^2_{\mathbb{Z}_-}(E). \end{cases} \tag{1.22}$$

As a result, we obtain the Hilbert space $D_-(N^*, \Gamma^*)$ generated by $\tilde{u}_n$, solutions of (1.22), besides $\|\tilde{u}_n\| = \|u_{n_1}\|_{l^2_{\mathbb{Z}_-}(E)}$.

The existence of the solution of the Cauchy problem (1.22) follows from Lem. 1.

Define the Hilbert space

$$\mathcal{H}_{N^*,\Gamma^*} = D_- \left(N^*, \Gamma^*\right) \oplus H \oplus D_+ \left(\tilde{N}^*, \tilde{\Gamma}^*\right), \tag{1.23}$$

in which the metric is induced by the norm of initial space $\mathcal{H} = D_- \oplus H \oplus D_+$ (1.4).

Define the operator function $\overset{+}{U}(n)$ for $n \in \hat{\mathbb{Z}}^2_+$ (1.16) in the space $\mathcal{H}_{N^*,\Gamma^*}$ (1.23), which acts on $\tilde{f} = \left(\tilde{u}_k, \tilde{h}, \tilde{v}_k\right) \in \mathcal{H}_{N^*,\Gamma^*}$ in the following way:

$$\overset{+}{U}(n)\tilde{f} = \tilde{f}(n) = \left(\tilde{u}_k(n), \tilde{h}(n), \tilde{v}(n)\right), \tag{1.24}$$

where $\tilde{v}_k(n) = P_{D_+\left(\tilde{N}^*,\tilde{\Gamma}^*\right)}\tilde{v}_{k+n}$ ($P_{D_+\left(\tilde{N}^*,\tilde{\Gamma}^*\right)}$ is an orthoprojector on $D_+\left(\tilde{N}^*, \tilde{\Gamma}^*\right)$); $\tilde{h}(n) = \tilde{y}_{(-1;0)}$, besides $\tilde{y}_k$ ($k \in \tilde{\mathbb{Z}}^2_-$) satisfies the Cauchy problem

$$\begin{cases} \partial_1 \tilde{y}_k = T_1^* \tilde{y}_k + \Psi^* \tilde{v}_{\tilde{k}}; \\ \partial_2 \tilde{y}_k = T_2^* \tilde{y}_k + \Psi^* N^* \tilde{v}_{\tilde{k}}; \\ \tilde{y}_{(-n_1,-n_2)} = h;\ k = (k_1, k_2) \in \tilde{\mathbb{Z}}^2_-; \end{cases} \tag{1.25}$$

besides $\tilde{k} = k + n$ and $(-n_1 \le k_1 \le -1;\ -n_2 \le k_2 \le 0)$; finally,

$$\tilde{u}_k(n) = \hat{u}_k + \tilde{u}_{k+n}, \tag{1.26}$$

and $\hat{u}_k = K^* \tilde{v}_{\tilde{k}} + \Phi^* \tilde{y}_k$, where $\tilde{y}_k$ is a solution of system (1.26).

It is clear that the semigroup $\overset{+}{U}(n)$ (1.24) is the isometric dilation [3] of the semigroup $T^*(n)$, where $T(n)$ has the form of (1.20).

Note that the dilations $U(n)$ (1.17) and $\overset{+}{U}(n)$ (1.24) are unitary linked, i.e., $U^*(n_1, 0)\, f = \overset{+}{U}(n_1, 0)\, f$ for all $f \in \mathcal{H}$ (1.4) and for all $n_1 \in \mathbb{Z}_+$, besides $U(n_1, 0)$ on $\mathcal{H}$ is a unitary semigroup.

2. Scattering Scheme with Many Parameters and Translational Models

I. As it is known [1, 9], the construction of translational (as well as functional) model of contraction T and its dilation U (1.5) follows naturally from the scattering scheme and from the properties of the wave operators $W_\pm$ and the scattering operator S.

In order to construct the wave operators $W_\pm$ in the case of many parameters it is necessary [4] to continue the vector functions from $l^2_{\mathbb{Z}}\left(\tilde{E}\right)$ and $l^2_{\mathbb{Z}}(E)$ from axis $\mathbb{Z}$ into domain $\mathbb{Z}^2$. Continue every function $u_{n_1} \in l^2_{\mathbb{Z}}(E)$ to the function u_n, where $n = (n_1, n_2) \in \mathbb{Z}^2$, using the Cauchy problem

$$\begin{cases} \tilde{\partial}_2 u_n = \left(N\tilde{\partial}_1 + \Gamma\right) u_n; & n \in \mathbb{Z}^2; \\ u_n|_{n_2=0} = u_{n_1} \in l^2_{\mathbb{Z}}(E); \end{cases} \tag{2.1}$$

besides $\|u_n\| = \|u_{n_1}\|_{l^2_{\mathbb{Z}}(E)}$. Note that this continuation into the lower half-plane ($n_2 \in \mathbb{Z}_-$), $u(n_1, n_2) \to u(n_1, n_2 - 1)$, has a recurrent nature and a continuation into the upper half-plane $u(n_1, n_2) \to u(n_1, n_2 + 1)$ may be carried out in a non-explicit way in the context of suppositions of Lem. 1.1. As a result, we obtain the Hilbert space $l^2_{N,\Gamma}(E)$ in which the norm is induced by the norm of initial data.

Define now the shift operator $V(p)$

$$V(p)u_n = u_{n-p}, \tag{2.2}$$

where $u_n \in l^2_{N,\Gamma}(E)$ for all $p \in \mathbb{Z}^2$. Obviously, the operator $V(p)$ (2.2) is isometric.

Knowing the perturbed $U(n)$ (1.17) and free $V(n)$ (2.2) operator semigroups, define the wave operator $W_-(n)$

$$W_-(k) = s - \lim_{n\to\infty} U(n,k) P_{D_-(N,\Gamma)} V(-n,-k) \tag{2.3}$$

for every fixed $k \in \mathbb{Z}_+$, where $P_{D_-(N,\Gamma)}$ is the orthoprojector of narrowing onto the component u_n^- from $l^2_{N,\Gamma}(E)$ obtained as a result of continuation into $\tilde{\mathbb{Z}}^2_-$ (1.7) from semiaxis $\mathbb{Z}_-$ using the Cauchy problem (2.1). It is obvious that $W_-(0) = W_-$, where the wave operator W_- corresponds with the dilation U (1.5) and the shift operator V in $l^2_{\mathbb{Z}}(E)$ [6]. Thus, $W_-(k)$ (2.3) is a natural continuation of the wave operator W_- onto the "k"th horizontal line in $\mathbb{Z}^2$ when $k \in \mathbb{Z}_+$.

Denote by $L^\infty_{0,k}$ the broken line in $\mathbb{Z}^2_+$ consisting of the two linear segments: the first one is a vertical segment connecting points $O = (0,0)$ and $(0,k)$, where $k \in \mathbb{Z}_+$, and the second segment is a horizontal half-line from point $(0,k)$ to (∞, k). Similarly, choose the broken line $\tilde{L}^{-1}_{-\infty,p}$ in $\tilde{\mathbb{Z}}^2_-$ (1.7) that also consists of the two linear segments, the first of which is a half-line from $(-\infty, -p)$ to point $(-1,-p)$, where $p \in \mathbb{Z}_+$, and the second one is a vertical segment from point $(-1,-p)$ to $(-1,0)$. In the space $\mathcal{H}_{N,\Gamma}$ (1.15), specify the following quadratic form:

$$\langle f\rangle^2_{\sigma(p,k)} = \langle \sigma u_n\rangle^2_{\tilde{L}^{-1}_{-\infty,p}} + \|h\|^2 + \langle \tilde{\sigma} v_n\rangle^2_{L^\infty_{0,k}}, \tag{2.4}$$

where corresponding σ and $\tilde{\sigma}$ in (2.4) are understood in the sense of (1.12) and (1.13).

Similarly to (2.4), in $l^2_{N,\Gamma}(E)$ specify the following σ-form:

$$\langle u_n\rangle^2_{\sigma(p,k)} = \left\langle \sigma u_n^-\right\rangle^2_{\tilde{L}^{-1}_{-\infty,p}} + \left\langle \sigma u_n^+\right\rangle_{L^\infty_{0,k}}, \tag{2.5}$$

where $u_n^{\pm}$ are the continuations of functions from $l^2_{\mathbb{Z}_\pm}(E)$ from semiaxes $\mathbb{Z}_\pm$, $n_2 = 0$, obtained by using the Cauchy problem (2.1).

Theorem 2.1 [4]. *The wave operator $W_-(k)$ (2.3) mapping $l^2_{N,\Gamma}(E)$ into the space $\mathcal{H}_{N,\Gamma}$ (1.15) exists for all $k \in \mathbb{Z}_+$, and it is an isometry*

$$\langle W_-(k)u_n\rangle^2_{\sigma(p,k)} = \langle u_n\rangle^2_{\sigma(p,k)} \tag{2.6}$$

in metrics (2.4), (2.5) for all $p \in \mathbb{Z}_+$. Moreover, the wave operator $W_-(k)$ (2.3) meets the conditions

$$\begin{array}{ll} 1) & U(1,s)W_-(k) = W_-(k+s)V(1,s); \\ 2) & W_-(k)P_{D_-(N,\Gamma)} = P_{D_-(N,\Gamma)} \end{array} \tag{2.7}$$

for all k, $s \in \mathbb{Z}_+$, where $P_{D_-(N,\Gamma)}$ is an orthoprojector onto $D_-(N,\Gamma)$.

II. Continue the vector functions v_{n_1} from $l^2_{\mathbb{Z}}\left(\tilde{E}\right)$ into domain $\mathbb{Z}^2$ using the Cauchy problem

$$\left\{ \begin{array}{ll} \tilde{\partial}_2 v_n = \left(\tilde{N}\tilde{\partial}_1 + \tilde{\Gamma}\right) v_n; & n = (n_1, n_2) \in \mathbb{Z}^2; \\ v_n|_{n_2=0} = v_{n_1} \in l^2_{\mathbb{Z}}\left(\tilde{E}\right). & \end{array} \right. \tag{2.8}$$

Denote the Hilbert space obtained in this way by $l_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$, besides $\|v_n\| = \|v_{n_1}\|_{l^2_{\mathbb{Z}}(\tilde{E})}$.

Similarly to $V(p)$ (2.2), introduce the shift operator

$$\tilde{V}(p)v_n = v_{n-p} \tag{2.9}$$

for all $p \in \mathbb{Z}^2$ and all $v_n \in l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$. Define the wave operator $W_+(p)$ from $\mathcal{H}_{N,\Gamma}$ into space $l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$

$$W_+(p) = s - \lim_{n\to\infty} \tilde{V}(-n,-p)P_{D_+(\tilde{N},\tilde{\Gamma})}U(n,p) \tag{2.10}$$

for all $p \in \mathbb{Z}_+$, where $U(n)$ has the form of (1.17). It is obvious that $W_+(0) = W_+^*$, where W_+ is the wave operator [1] corresponding to U (1.5) and to shift $\tilde{V}$ in $l^2_{\mathbb{Z}}\left(\tilde{E}\right)$.

Theorem 2.2 [4]. *For all $p \in \mathbb{Z}_+$, the wave operator $W_+(p)$ (2.11) acting from space $\mathcal{H}_{N,\Gamma}$ into $l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$ exists and satisfies the relations*

$$\begin{array}{ll} 1) & W_+(p)U(1,s) = \tilde{V}(1,s)W_+(p+s); \\ 2) & W_+(p)P_{D_+(\tilde{N},\tilde{\Gamma})} = P_{D_+(\tilde{N},\tilde{\Gamma})} \end{array} \tag{2.11}$$

for all p, $s \in \mathbb{Z}_+$, *where* $P_{D_+(\tilde{N},\tilde{\Gamma})}$ *is an orthoprojector onto* $D_+\left(\tilde{N},\tilde{\Gamma}\right)$.

Knowing the wave operators $W_-(k)$ (2.3) and $W_+(p)$ (2.10), define the scattering operator in a traditional way [1, 4]:

$$S(p,k) = W_+(p)W_-(k) \tag{2.12}$$

for all p, $k \in \mathbb{Z}_+$. It is obvious that when $p = k = 0$, we have $S(0,0) = S$, where S is the standard scattering operator, $S = W_+^* W_-$, for the dilation U (1.5) [1].

Theorem 2.3 [4]. *The scattering operator* $S(p,k)$ *(2.13) represents the bounded operator from* $l^2_{N,\Gamma}(E)$ *into* $l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$, *besides*

$$\begin{array}{ll} 1) & S(p,k)V(1,q) = \tilde{V}(1,q)S(p+q,k-q); \\ 2) & S(p,k)P_- l^2_{N,\Gamma}(E) \subseteq P_- l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right) \end{array} \tag{2.13}$$

for all p, k, $q \in \mathbb{Z}_+$, $0 \le q \le k$, *where* P_- *is the narrowing orthoprojector onto solutions of the Cauchy problems (2.1) and (2.9) with the initial data on semiaxis* $\mathbb{Z}_-$ *when* $n_2 = 0$.

III. Following [4], consider the nonnegative operator function $W_{p,k}$

$$W_{p,k} = \begin{bmatrix} W_+(p)W_+^*(p) & S(p,k) \\ S^*(p,k) & W_-^*(k)W_-(k) \end{bmatrix} \tag{2.14}$$

to define the Hilbert space

$$l^2\left(W_{p,k}\right) = \left\{ g_n = \begin{pmatrix} v_n \\ u_n \end{pmatrix} : \langle W_{p,k} g_n, g_n \rangle_{l^2} < \infty \right\}, \tag{2.15}$$

where $u_n \in l^2_{N,\Gamma}(E)$, $v_n \in l^2_{\tilde{N},\Gamma}\left(\tilde{E}\right)$.

Let

$$\begin{array}{l} W'_{p,0} = \begin{bmatrix} \tilde{V}(1,p)W_+(p)W_+^*(p)\tilde{V}^*(1,p) & S(0,p) \\ S^*(0,p) & I \end{bmatrix}; \\ \hat{V}(1,p) = \begin{bmatrix} \tilde{V}^*(-1,-p) & 0 \\ 0 & V(1,p) \end{bmatrix}. \end{array} \tag{2.16}$$

As it follows from [9], the operator

$$\hat{U}(1,p)g_n = \hat{V}(1,p)g_n \tag{2.17}$$

acts from the Hilbert space

$$l^2\left(W'_{p,0}\right) = \left\{ g_n = \begin{pmatrix} v_n \\ u_n \end{pmatrix} : \left\langle W'_{p,0} g_n, g_n \right\rangle_{l^2} < \infty \right\} \tag{2.15'}$$

into the space $l^2(W_{p,0})$ (2.15).

Denote by $\hat{H}_p$ the Hilbert space

$$\hat{H}_p = l^2(W_{p,0}) \ominus \begin{pmatrix} P_+ l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right) \\ P_- l^2_{N,\Gamma}(E) \end{pmatrix}, \tag{2.18}$$

where $P_\pm$ are orthoprojectors onto solutions of the Cauchy problems (2.1), (2.8) with the initial data on $\mathbb{Z}_\pm$. Consider also

$$\hat{H}'_p = l^2\left(W'_{p,0}\right) \ominus \begin{pmatrix} \tilde{V}^*(-1,-p) P_+ l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right) \\ V(1,p) P_- l^2_{N,\Gamma}(E) \end{pmatrix}. \tag{2.18$'$}$$

The spaces $\hat{H}_p$ (2.18) and $\hat{H}'_p$ (2.18$'$) are isomorphic one to another, besides, as it is easily seen, the operator $R_p : \hat{H}_p \to \hat{H}'_p$ defining this isomorphism has the form

$$R_p = P_{\hat{H}'_p} \begin{bmatrix} \tilde{V}^*(1,p) & 0 \\ 0 & V(-1,-p) \end{bmatrix} P_{\hat{H}_p}, \tag{2.19}$$

where $P_{\hat{H}_p}$ and $P_{\hat{H}'_p}$ are orthoprojectors onto $\hat{H}_p$ (2.18) and $\hat{H}'_p$ (2.18$'$), respectively. Specify the operators $\hat{T}_1$ and $\hat{T}(1,p) = \hat{T}_1 \hat{T}_2^p$, $p \in \mathbb{Z}_+$,

$$\left(\hat{T}_1 f\right)_n = P_{\hat{H}_p} f_{n-(1,0)}; \quad \left(\hat{T}(1,p) f\right)_n = P_{\hat{H}_p} \hat{V}(1,p) \left(R_p f\right)_n \tag{2.20}$$

for all $f_n \in \hat{H}_p$ (2.18). Note that the operator $\hat{T}_1$ has the same form (2.20) in all spaces $\hat{H}_p$ (2.28).

Theorem 2.4 [4]. *Consider the simple commutative unitary extension* $(V_s, \overset{+}{V}_s)$ *(2.1) corresponding to the commutative operator system* $\{T_1, T_2\}$ *from the class* $C(T_1)$ *(1.3) and let the suppositions of Lem. 1.1 take place, besides* $\dim E = dim\tilde{E} < \infty$. *Then the isometric dilation* $U(1,p)$ *(1.17),* $p \in \mathbb{Z}_+$, *acting in the Hilbert space* $\mathcal{H}_{N,\Gamma}$ *(1.15) is unitary equivalent to the operator* $\hat{U}(1,p)$ *(2.17) mapping the space* $l^2\left(W'_{p,0}\right)$ *(2.15$'$) into* $l^2(W_{p,0})$ *(2.15). Moreover, the operators* T_1 *and* $T(1,p) = T_1 T_2^p$ *(1.21) specified in* H *are unitary equivalent to the shift operator* $\hat{T}_1$ *(2.20) and to the operator* $\hat{T}(1,p)$ *(2.20).*

A similar translational model of dilation $\overset{+}{U}(n)$ (1.24) and semigroup $T^*(n)$ (1.20) is listed in [4].

3. Functional Models

I. In order to construct the functional models of dilations $U(n)$ (1.17) and $\overset{+}{U}(n)$ (1.24), it is necessary to realize the Fourier transformation of translational models from Sect. 2. The Fourier transformation $\mathcal{F}$

$$\mathcal{F}(u_k) = \sum_{k\in\mathbb{Z}} u_k \xi^k = u(\xi), \quad u_k \in l^2_{\mathbb{Z}}(E), \tag{3.1}$$

specifies the isomorphism between $l^2_{\mathbb{Z}}(E)$ and the Hilbert space $L^2_{\mathbb{T}}(E)$ [1, 9].

Realize the Fourier transformation $\mathcal{F}$ (3.1) by variable n_1 of every vector function u_n from the space $l^2_{N,\Gamma}(E)$, $n = (n_1, n_2) \in \mathbb{Z}^2$. Then we obtain (see the Cauchy problem (2.1)) the family of functions $u(\xi, n_2)$ specified on every n_2-th horizontal line ($n_2 \in \mathbb{Z}$), besides the transition from n_2 to $n_2 - 1$ is specified by multiplication by the linear pencil of operators

$$u(\xi, n_2 - 1) = (N\xi + \Gamma) u(\xi, n_2). \tag{3.2}$$

Note that a corresponding continuation into half-plane $n_2 \in \mathbb{Z}_+$ may be carried out in the context of suppositions of Lem. 1.1 when $\dim E < \infty$. As a result, we obtain the Hilbert space of functions $u(\xi, n_2)$, for which (3.2) takes place, besides $u(\xi) = u(\xi, 0) \in L^2_{\mathbb{T}}(E)$. We denote this space by $L^2_{\mathbb{T}}(N, \Gamma, E)$. It is obvious that the shift operator $V(p)$ (2.2), as a result of the Fourier transformation $\mathcal{F}$ (3.1) in space $L^2_{\mathbb{T}}(N, \Gamma, E)$, acts by multiplication

$$V(p)u(\xi) = \xi^{p_1}(N\xi + \Gamma)^{p_2} u(\xi), \tag{3.3}$$

where $u(\xi) = u(\xi, 0)$ and $p = (p_1, p_2) \in \mathbb{Z}^2$. Similarly, the Fourier transformation $\mathcal{F}$ (3.1) of space $l^2_{\mathbb{Z}}\left(\tilde{E}\right)$ leads us to the Hilbert space $L^2_{\mathbb{T}}\left(\tilde{E}\right)$. The Fourier transformation $\mathcal{F}$ by the first variable n_1 of every function $v_n = v_{(n_1,n_2)}$ from $l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$ gives us the family of $\tilde{E}$-valued functions $v(\xi, n_2)$, for which

$$v(\xi, n_2 - 1) = \left(\tilde{N}\xi + \tilde{\Gamma}\right) v(\xi, n_2) \tag{3.4}$$

takes place in view of the Cauchy problem (2.8). The obtained space of functions $v(\xi, n_2)$, where $v(\xi) = v(\xi, 0) \in L^2_{\mathbb{T}}\left(\tilde{E}\right)$, we denote by $L^2_{\mathbb{T}}\left(\tilde{N}, \tilde{\Gamma}, \tilde{E}\right)$. As in the previous case, the continuation by rule (3.5), when $n_+2 \in \mathbb{Z}_+$, is possible when the suppositions of Lem. 1.1 are met and $\dim \tilde{E} < \infty$. The translation operator $\tilde{V}(p)$ (2.9) in the Hilbert space $L^2_{\mathbb{T}}\left(\tilde{N}, \tilde{\Gamma}, \tilde{E}\right)$ is realized by multiplication operator

$$\tilde{V}(p)v(\xi) = \xi^{p_1}\left(\tilde{N}\xi + \tilde{\Gamma}\right)^{p_1} v(\xi), \tag{3.5}$$

where $v(\xi) = v(\xi, 0)$ and $p = (p_1, p_2) \in \mathbb{Z}^2$.

II. The translational invariance (2.13) of the operator $S(p,k)$ (2.11) signifies that the Fourier image of the scattering operator $S(p,k)$ represents the operator of multiplication by vector function. In particular, $\mathcal{F}S(0,0)u_k = S(\xi)u(\xi)$, where $u(\xi) = \mathcal{F}(u_k)$ (3.1) and $S(\xi) = K + \Psi(\xi I - T_1)^{-1}\Phi$ is the characteristic function of extension V_1 (1.1) of the operator T_1. It follows from relation 1) (2.13) for the operator $S(p,k)$ that it is necessary to find the Fourier image of operator $S(p,0)$ (or of $S(0,p)$, in view of 1) (2.13)) for all $p \in \mathbb{Z}_+$. Further, taking into account the translational invariance of operator $S(p,0)$, it is obvious that it is sufficient to calculate how $S(p,0)$ acts on the vector function $u_k^0 = u\delta_{k,0}$, where u is an arbitrary vector from E, and $\delta_{k,0}$ is the Kronecker symbol. For simplicity, consider the case p=1, then it follows from (2.3) and from (2.10) that

$$v_n^m = \tilde{V}(-m,-1)P_{D_+(\tilde{N},\tilde{\Gamma})}U(2m,1)P_{D_-(N,\Gamma)}V(-m,0)u_k^0 \to S(1,0)u_k^0$$

when $m \to \infty$, $n \in \mathbb{Z}^2$. Elementary calculations show that the vector function v_n^m is given by

$$v_{(n_1,0)}^m = \left(...,0,\Psi T_1^{m-1}\Phi u,...,\Psi T_1\Phi u,\Psi\Phi u,\boxed{Ku},0,...\right);$$

$$v_{(n_1,-1)}^m = \left(...,0,\Psi T_1^{m-1}T_2\Phi u,...,\Psi T_1 T_2\Phi u,\Psi T_2\Phi u,\boxed{(K\Gamma + \Psi\Phi N)u},KNu,0,...\right),$$

where the frame signifies the element corresponding to the null index, $n_1 = 0$. After the limit process, when $n \to \infty$ and the Fourier transformation is $\mathcal{F}$ (3.1), we obtain that the components $v(\xi, n_2)$ are given by

$$v(\xi,0) = S(\xi)u;$$
$$v(\xi,-1) = \left\{KN\xi + K\Gamma + \Psi\Phi N + \Psi(\xi - T_1)^{-1}T_2\Phi\right\}u.$$

Using now 3) (1.2), we obtain that

$$v(\xi,-1) = S(\xi)(N\xi + \Gamma)u. \tag{3.6}$$

Taking into account colligation relations 4), 5) (1.2), we can rewrite the equality (3.6) in the following way:

$$v(\xi,-1) = \left(\tilde{N}\xi + \tilde{\Gamma}\right)S(\xi)u. \tag{3.7}$$

Define the "kth" characteristic function $S(\xi,k)$ using the formula

$$S(\xi,k) = S(\xi)(N\xi + \Gamma)^k, \quad k \in \mathbb{Z}_+, \tag{3.8}$$

where $S(\xi) = K + \Psi(\xi I - T_1)\Phi$ and $S(\xi,0) = S(\xi)$.

Theorem 3.1. *Let $u_k \in l^2_{\mathbb{Z}}(E)$ and $u(\xi) = \mathcal{F}(u_k)$ (3.1). Then the Fourier transformation $\mathcal{F}$ applied to the vector function $v = S(p,0)u$ represents the family of $\tilde{E}$-valued functions $v(\xi,-k)$, where $0 \le k \le p$, $k \in \mathbb{Z}_+$, such that*

$$v(\xi,-k) = S(\xi,k)u(\xi), \tag{3.9}$$

besides the functions $S(\xi,k)$ are given by (3.8), $0 \le k \le p$, where $S(\xi,0) = S(\xi) = K + \Psi(\xi I - T_1)^{-1}\Phi$ is the characteristic function of extension V_1 (1.1) corresponding to operator T_1.

Thus the Fourier transformation $\mathcal{F}$ of operator $S(p,0)$ leads us to the operator of multiplication by characteristic function $S(\xi)$ of the family of functions $u(\xi,n_2)$ from the space $L^2_{\mathbb{T}}(N,\Gamma,E)$ when $n_2 \in \mathbb{Z}_- \cup \{0\}$.

III. In order to find a Fourier image of the weight function $W_{p,0}$ (2.14), it is necessary to calculate the Fourier transformation of the operator $W_+(p)W_+^*(p)$ which is also the operator of multiplication by operator function. It follows from the definition (2.10) of the wave operator $W_+(p)$ that $W(n,p) \to W_+(p)W_+^*(p)$ when $n \to \infty$, where

$$W(n,p) = \tilde{V}(-n,-p)P_{D_+(\tilde{N},\tilde{\Gamma})}U(n,p)U^*(n,p)P_{D_+(\tilde{N},\tilde{\Gamma})}\tilde{V}^*(-n,-p). \tag{3.10}$$

Using the unitary properties of $U(n,0)$ and $\tilde{V}(n,0)$, $n \in \mathbb{Z}$, it is easy to ascertain that

$$W(n+1,p) = \tilde{V}(-n,0)W(1,p)\tilde{V}(n,0). \tag{3.11}$$

Therefore, it is sufficient to calculate how the operator $W(1,p)$ acts. For simplicity, conduct calculations for the case of $p = 2$. Let $f = (u_k, h, v_k) \in \mathcal{H}_{N,\Gamma}$ (1.15) then, using the form of U (1.17), it is easy to show that

$$\tilde{V}(-1,-2)P_{D_+(\tilde{N},\tilde{\Gamma})}U(1,2)f = \hat{v}_k \oplus P_+v_k, \tag{3.12}$$

where P_+, as usually, is the orthoprojector in $l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$ on the subspace of solutions of the Cauchy problem (1.9) with the initial data on semiaxis $\mathbb{Z}_+$, and the vector function $\hat{v}_k$ from $\tilde{E}$ is defined at points $(-1,0)$, $(-1,-1)$, $(-1,-2)$ in the following way:

$$\begin{aligned} &\hat{v}_{-1,0} = \Psi h + Ku_{1,0}; \quad \hat{v}_{-1,-1} = \Psi\left(T_2h + \Phi N_{-1,0}\right) + Ku_{-1,-1}; \\ &\hat{v}_{-1,-2} = \Psi\left\{T_2\left(T_2h + \Phi N_{-1,0}\right) + \Phi Nu_{-1,-1}\right\} + Ku_{-1,-2}. \end{aligned} \tag{3.13}$$

Make use of the fact that the function u_k is a solution of the Cauchy problem (1.8). Then, taking into account relations 3)–5) (1.2), we obtain that it is possible to write down the relations for the components (3.13), where $k = 0, -1, -2$, in the following form:

$$\begin{bmatrix} \hat{v}_{-1,0} \\ \hat{v}_{-1,-1} \\ \hat{v}_{-1,-2} \end{bmatrix} =$$

$$= \begin{bmatrix} I & 0 & 0 \\ \tilde{\Gamma} & \tilde{N} & 0 \\ \tilde{\Gamma}^2 & \tilde{N}\tilde{\Gamma} + \tilde{\Gamma}\tilde{N} & \tilde{N}^2 \end{bmatrix} \begin{bmatrix} \Psi h + K u_{-1,0} \\ \Psi T_1 h + \Psi\Phi u_{-1,0} + K u_{-2,0} \\ \Psi T_1^2 h + \Psi T_1 \Phi u_{-1,0} + \Psi\Phi u_{-2,0} + K u_{-3,0} \end{bmatrix}. \tag{3.14}$$

Note that the right-hand member of equality (3.14) is expressed in the terms of operator T_1 and external parameters of extension (1.1), and, moreover, the coefficients before $u_{-1,k}$, $k = 0, -1, -2$, coincide with the corresponding coefficients of the Laurent factorization of characteristic function $S(\xi) = K + \Psi\left(\xi I - T_1\right)^{-1}\Phi$ (1.7) of the operator T_1. Introduce into examination the matrices

$$\tilde{L}_2 = \begin{bmatrix} I & 0 & 0 \\ \tilde{\Gamma} & \tilde{N} & 0 \\ \tilde{\Gamma}^2 & \tilde{\Gamma}\tilde{N} + \tilde{N}\tilde{\Gamma} & \tilde{N}^2 \end{bmatrix};$$
$$Q_2 = \begin{bmatrix} \Psi & 0 & 0 \\ \Psi T_1 & 0 & 0 \\ \Psi T_1^2 & 0 & 0 \end{bmatrix}; \quad R_2 = \begin{bmatrix} K & 0 & 0 \\ \Psi\Phi & K & 0 \\ \Psi T_1 \Phi & \Psi\Phi & K \end{bmatrix}. \tag{3.15}$$

Then it follows from (3.14) that the operator $W(1,2)$ (3.10) is given by

$$W(1,2) = P_{-1}\tilde{L}_2\left\{Q_2 Q_2^* + R_2 R_2^*\right\}\tilde{L}_2^* P_{-1} \oplus P_{D_+\left(\tilde{N},\tilde{\Gamma}\right)}, \tag{3.16}$$

where P_{-1} is the orthoprojector of narrowing on the vertical line $n_1 = -1$ of grid $\mathbb{Z}^2$ or the operator of multiplication by the Kronecker symbol $\delta_{n_1,-1}$. If one makes use of the relations $\Psi\Psi^* + KK^* = I$, $\Psi T_1^* + K^* = 0$ and $T_1 T_1^* + \Phi\Phi^* = I$ that follow from condition 1) (1.2), then it is easy to show that

$$Q_2 Q_2^* + R_2 R_2^* = I. \tag{3.17}$$

Therefore, we finally obtain that

$$W(1,2) = P_{-1}\tilde{L}_2\tilde{L}_2^* P_{-1} \oplus P_{D_+\left(\tilde{N},\tilde{\Gamma}\right)}. \tag{3.18}$$

IV. In order to find the Fourier transformation of operator $W(1,2)$ (3.18), calculate the Fourier image of matrix $\tilde{L}_2$ (3.15). Let $v(\xi) = v(\xi,0) = \sum_{-\infty}^{-1}\xi^k v_k \in L^2_{\mathbb{T}}\left(\tilde{E}\right)$, further construct the family of functions $v\left(\xi, n_2\right)$ from space $L^2_{\mathbb{T}}\left(\tilde{N},\tilde{\Gamma},\tilde{E}\right)$ by rule (3.5)

$$v(\xi,-k) = \left(\tilde{N}\xi + \tilde{\Gamma}\right)^k v(\xi), \quad k = 0, 1, 2. \tag{3.19}$$

It is easy to make sure that the coefficients before $\overline{\xi}$ in the family of functions $v(\xi,-k)$ (3.19), where $k = 0$, 1, 2, correspondingly are equal to $v_{-1,0}$, $\tilde{N}v_{-2,0} +$

$\tilde{\Gamma} v_{-1,0}$, $\tilde{N}^2 v_{-3,0} + \left(\tilde{N}\tilde{\Gamma} + \tilde{\Gamma}\tilde{N}\right) v_{-2,0} + \tilde{\Gamma}^2 v_{-1,0}$, which signifies the application of matrix $\tilde{L}_2$ (3.15) to the vector column created by elements $v_{-1,0}$, $v_{-2,0}$, $v_{-3,0}$. Therefore the Fourier transformation $\mathcal{F}$ (3.1) of the operator $P_{-1}\tilde{L}_2\tilde{L}_2^* P_{-1}$ is given by

$$P_{-1}\begin{bmatrix} I & 0 & 0 \\ \tilde{N}\xi + \tilde{\Gamma} & 0 & 0 \\ \left(\tilde{N}\xi + \tilde{\Gamma}\right)^2 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} I & \tilde{N}^*\overline{\xi} + \tilde{\Gamma}^* & \left(\tilde{N}^*\overline{\xi} + \tilde{\Gamma}^*\right)^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P_{-1} \begin{bmatrix} v(\xi) \\ v(\xi,-1) \\ v(\xi,-2) \end{bmatrix}, \tag{3.20}$$

where P_{-1} is the operator of projection on the subspace $\left\{\overline{\xi} v\right\}$, $v \in \tilde{E}$, and the functions $v(\xi,-k)$ are constructed by rule (3.19), $k = 0, 1, 2$. Taking into account the projector P_{-1}, after elementary calculations it is easy to see that the relation (3.20) is equal to

$$\tilde{L}_2\tilde{L}_2^* P_{-1}\begin{bmatrix} v(\xi) \\ v(\xi,-1) \\ v(\xi,-2) \end{bmatrix} = W_2 P_{-1}\begin{bmatrix} v(\xi) \\ v(\xi,-1) \\ v(\xi,-2) \end{bmatrix}.$$

Thus, it follows from (3.10), (3.11) and (3.18) that the Fourier transformation of the operator $W_+(2)W_+^*(2)$ is given by

$$\mathcal{F}\left(W_+(2)W_+^*(2)v_n\right) = \left\{I - P_-\left(I - W_2\right)P_-\right\} v\left(\xi, n_2\right), \tag{3.21}$$

where $v_n \in l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$, $v\left(\xi,n_2\right) \in L^2_{\mathbb{T}}\left(\tilde{N},\tilde{\Gamma},\tilde{E}\right)$, $W_2 = \tilde{L}_2\tilde{L}_2^*$, and P_- is the orthoprojector on the subspace of functions of the type $\sum\limits_{-\infty}^{-1}\xi^k v_k$, $v_k \in \tilde{E}$. To formulate the overall result for all $p \in \mathbb{Z}_+$, define the constant matrix

$$W_p = P_0 \begin{bmatrix} I & 0 & \cdots & 0 \\ \tilde{N}\xi + \tilde{\Gamma} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \left(\tilde{N}\xi + \tilde{\Gamma}\right)^p & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} I & \tilde{N}^*\overline{\xi} + \tilde{\Gamma}^* & \cdots & \left(\tilde{N}^*\overline{\xi} + \tilde{\Gamma}\right)^p \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.22}$$

where P_0 is the operator of narrowing of every component of multiplication (3.22) of matrix $(p+1) \times (p+1)$ on the elements corresponding ξ^0.

Theorem 3.2. *The Fourier transformation $\mathcal{F}$ (3.1) of the operator $W_+(p)W_+^*(p)$, where the operator $W_+(p)$ is given by (2.10), is the multiplication* *by constant matrix,*

$$\mathcal{F}\left(W_+(p)W_+^*(p)v_n\right) = \left\{I - P_-\left(I - W_p\right)P_-\right\} v\left(\xi, n_2\right), \tag{3.23}$$

besides W_p *is given by (3.22),* $v(\xi, n_2) = \mathcal{F}(v_n) \in L^2_{\mathbb{T}}\left(\tilde{N}, \tilde{\Gamma}, \tilde{E}\right)$, *where* $v_n \in l^2_{\tilde{N},\tilde{\Gamma}}\left(\tilde{E}\right)$ *and* P_- *is the orthoprojector in* $L^2_{\mathbb{T}}\left(\tilde{N}, \tilde{\Gamma}, \tilde{E}\right)$ *on the subspace of functions* $v(\xi, n_2)$ *such that* $v(\xi, 0)$ *is factorized into the series by powers* $\left\{\xi^k\right\}_{k\in\mathbb{Z}_-}$, *besides* $v(\xi, n_2)$ *are obtained from* $v(\xi, 0)$ *by rule (3.5).*

V. It follows from Ths. 3.1 and 3.2 that the operator weight $W_{p,0}$ (2.14) after the Fourier transformation $\mathcal{F}$ (3.1) is the operator of multiplication by function

$$W(p, \xi) = \begin{bmatrix} I - P_-\left(I - W_p\right)P_- & S(\xi) \\ S^*(\xi) & I \end{bmatrix}, \tag{3.24}$$

where W_p is a constant matrix of the type (3.22), and $S(\xi)$ is the characteristic function of extension V_1. After this, it is obvious that the space $l^2\left(W_{p,0}\right)$ (2.15), as a result of the Fourier transformation $\mathcal{F}$ (3.1), is given by

$$L^2_{\mathbb{T}}(W(p,\xi)) = \left\{ g(\xi) = \begin{pmatrix} v(\xi) \\ u(\xi) \end{pmatrix} : \int\limits_0^{2\pi} \langle W(p,\xi)g(\xi), g(\xi)\rangle \frac{d\xi}{2\pi i \xi} < \infty \right\}, \tag{3.25}$$

where $u(\xi) = u(\xi, 0) \in L^2_{\mathbb{T}}(E)$, and is continued to the family of functions $u(\xi, n_2)$ from $L^2_{\mathbb{T}}(N, F, E)$ by rule (3.2), and $v(\xi) = v(\xi, 0) \in L\left(\tilde{E}\right)$ and it also has a continuation to the family $v(\xi, n_2)$ from $L^2_{\mathbb{T}}\left(\tilde{N}, \tilde{\Gamma}, \tilde{E}\right)$ by formula (3.5). Using again Ths. 3.1 and 3.2, it is easy to ascertain that the Fourier image of operator $W'_{p,0}$ (2.15′) is the operator of multiplication by function

$$W'(p, \xi) = \begin{bmatrix} \left(\tilde{N}\xi + \tilde{\Gamma}\right)^p \left\{I - P_-\left(I - W_p\right)P_-\right\}\left(\tilde{N}^*\overline{\xi} + \tilde{\Gamma}^*\right)^p & S(\xi) \\ S^*(\xi) & I \end{bmatrix}. \tag{3.26}$$

Therefore, the space $l^2\left(W'_{p,0}\right)$ (2.15′), after the Fourier transformation $\mathcal{F}$ (3.1), is given by

$$L^2_{\mathbb{T}}\left(W'(p,\xi)\right) = \left\{ g(\xi) = \begin{pmatrix} v(\xi) \\ u(\xi) \end{pmatrix} : \int\limits_0^{2\pi} \left\langle W'(p,\xi)g(\xi), g(\xi)\right\rangle \frac{d\xi}{2\pi i \xi} < \infty \right\}, \tag{3.25′}$$

where $u(\xi)$ and $v(\xi)$ have the same sense as in the definition of space $L^2_{\mathbb{T}}(W(p,\xi))$ (3.25).

In view of (3.3) and (3.5), it follows from (2.17) that the dilations $U(1,0)$ and $U(1,p)$ are the multiplication operators

$$\tilde{U}(1,0)g(\xi) = \xi g(\xi);$$
$$\tilde{U}(1,p)g(\xi) = \xi \begin{bmatrix} \left(\tilde{N}^*\overline{\xi} + \tilde{\Gamma}^*\right)^{-p} & 0 \\ 0 & (N\xi + \Gamma)^p \end{bmatrix} g(\xi), \tag{3.27}$$

where $p \in \mathbb{Z}_+$ and $g(\xi) \in L^2_{\mathbb{T}}(W'(p,\xi))$. It is easy to see that the model space $\hat{H}_p$ (2.18) after the Fourier transformation is equal to

$$\tilde{H}_p = L^2_{\mathbb{T}}(W(p,\xi)) \ominus \begin{pmatrix} H^2_+\left(\tilde{N},\tilde{\Gamma},\tilde{E}\right) \\ H^2_-(N,\Gamma,E) \end{pmatrix}, \tag{3.28}$$

where the Hardy subspaces $H^2_-(N,\Gamma,E)$ and $H^2_+\left(\tilde{N},\tilde{\Gamma},\tilde{E}\right)$ are obtained from ordinary Hardy classes $H^2_-(E)$ and $H^2_+\left(\tilde{E}\right)$ corresponding to domains $\mathbb{D}_- = \{z \in \mathbb{C} : |z| > 1\}$ and $\mathbb{D}_+ = \{z \in \mathbb{C} : |z| < 1\}$ using the rules (3.2) and (3.5), respectively.

O b s e r v a t i o n 1. Note that the Hardy space $H^2_-(N,\Gamma,E)$ contains the functions that are not holomorphic in $\mathbb{D}_-$. Really, every function $u(\xi,-n_2) = (N\xi+\Gamma)^{n_2}u(\xi)$, where $u(\xi) \in H^2_-(E)$ and $n_2 \in \mathbb{Z}_+$, is factorized into the Fourier series by powers $\{\xi^k\}$ when $k \in (\mathbb{Z}_- + n_2 - 1)$.

Similarly, the space $\hat{H}'_p$ (2.18′) after the Fourier transformation $\mathcal{F}$ (3.1) is given by

$$\tilde{H}'_p = L^2_{\mathbb{T}}\left(W'(p,\xi)\right) \ominus \begin{pmatrix} \xi\left(\tilde{N}^*\overline{\xi} + \tilde{\Gamma}^*\right) H^2_+\left(\tilde{N},\tilde{\Gamma},\tilde{E}\right) \\ \xi(N\xi+\Gamma)^p H^2_-(N,\Gamma,E) \end{pmatrix}, \tag{3.28′}$$

where the weight $W'(p,\xi)$ is given by formula (3.26). The isomorphism $\tilde{R}_p : \tilde{H}_p \to \tilde{H}'_p$ after the Fourier transformation of the operator R_p (2.19) represents

$$\tilde{R}_p = P_{\tilde{H}'_p} \begin{bmatrix} \overline{\xi}\left(\tilde{N}^*\overline{\xi} + \tilde{\Gamma}^*\right)^p & 0 \\ 0 & \overline{\xi}(N\xi+\Gamma)^{-p} \end{bmatrix} P_{\tilde{H}_p}, \tag{3.29}$$

where $P_{\tilde{H}_p}$ and $P_{\tilde{H}'_p}$ are the orthoprojectors on $\tilde{H}_p$ (3.28) and $\tilde{H}'_p$ (3.28′), respectively. Finally, the operators T_1 and $T(1,p) = T_1T_2^p$ in space (3.28), in view of (3.27), act in the following way:

$$\left(\tilde{T}_1 f\right)(\xi) = P_{\tilde{H}_p}\xi f(\xi);$$
$$\left(\tilde{T}(1,p)f\right)(\xi) = P_{\tilde{H}}\xi \begin{bmatrix} \left(\tilde{N}^*\overline{\xi} + \tilde{\Gamma}^*\right)^{-p} & 0 \\ 0 & (N\xi+\Gamma)^p \end{bmatrix} \left(\tilde{R}_p f\right)(\xi), \tag{3.30}$$

where $f(\xi) \in \tilde{H}_p$ (3.28), and $P_{\tilde{H}_p}$ is the orthoprojector on $\tilde{H}_p$ (3.28), besides $\tilde{R}_p$ is given by (3.29). From this it follows immediately that the initial operator system $\{T_1, T_2\}$, given in H in space $\tilde{H}_1$ (3.28), will represent

$$\left(\tilde{T}_1 f\right)(\xi) = P_{\tilde{H}_1} \xi f(\xi);$$

$$\left(\tilde{T}_2 f\right)(\xi) = P_{\tilde{H}_1} \begin{bmatrix} \left(\tilde{N}^* \overline{\xi} + \tilde{\Gamma}^*\right)^{-1} & 0 \\ 0 & N\xi + \Gamma \end{bmatrix} \left(\tilde{R}_1 f\right)(\xi), \tag{3.31}$$

where $f(\xi) \in \tilde{H}_1$ (3.28).

Theorem 3. *Consider the simple [2, 3] commutative unitary extension* $(V_s, \overset{+}{V}_s)$ *(1.1) corresponding to the commutative operator system* $\{T_1, T_2\}$ *from the class* $C(T_1)$ *(1.3), and let the suppositions of Lem. 1.1 be met, besides* $\dim E = \dim \tilde{E} < \infty$*. Then the isometric dilation* $U(1,p)$ *(1.17) acting in the Hilbert space* $\mathcal{H}_{N,\Gamma}$ *(1.15) is unitary equivalent to the functional model* $\tilde{U}(1,0)$ *(3.27), when* $p = 0$*, in* $L^2_{\mathbb{T}}(W'(p,\xi))$ *(3.25′) and to the operator* $\tilde{U}(1,p)$ *(3.27), when* $p \in \mathbb{N}$*, mapping the space* $L^2_{\mathbb{T}}(W'(p,\xi))$ *(3.25′) into the space* $L^2_{\mathbb{T}}(W(p,\xi))$ *(3.25). Moreover, the operators* T_1 *and* $T(1,p) = T_1 T_2^p$ *(1.20) given in* H *are unitary equivalent to the functional model* $\tilde{T}_1$ *(3.30) in* $\tilde{H}_p$ *for all* $p \in \mathbb{Z}_+$ *and to the operator* $\tilde{T}_1(1,p)$ *(3.30) in the concrete model space* $\tilde{H}_p$ *(3.28) when* $p \in \mathbb{N}$.

VI. We now turn to the dual situation corresponding to the dilation $\overset{+}{U}(n)$ (1.24) in $\mathcal{H}_{N^*,\Gamma^*}$. We list the main results concerning this case without proving.

Define the constant matrix $\tilde{W}_p$ for all $p \in \mathbb{Z}_+$

$$\tilde{W}_p = P_0 \begin{bmatrix} I & 0 & \cdots & 0 \\ N^*\overline{\xi} + \Gamma^* & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \left(N^*\overline{\xi} + \Gamma^*\right)^p & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} I & N\xi + \Gamma & \cdots & (N\xi + \Gamma)^p \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.32}$$

where P_0 is the operator of narrowing on the components corresponding to ξ^0.

Consider the weight operator function

$$\tilde{W}(p,\xi) = \begin{bmatrix} I & S(\xi) \\ S^*(\xi) & I - P_+ \left(I - \tilde{W}_p\right) P_+ \end{bmatrix}, \tag{3.33}$$

where the constant matrix $\tilde{W}_p$ is given by (3.32). Specify the Hilbert space

$$L^2_{\mathbb{T}}\left(\tilde{W}(p,\xi)\right) = \left\{ g(\xi) = \begin{pmatrix} v(\xi) \\ u(\xi) \end{pmatrix} : \int_0^{2\pi} \left\langle \tilde{W}(p,\xi) g(\xi), g(\xi) \right\rangle \frac{d\xi}{2\pi i \xi} < \infty \right\}, \tag{3.34}$$

where $u(\xi) \in L^2_{\mathbb{T}}(E)$, $v(\xi) \in L^2_{\mathbb{T}}\left(\tilde{E}\right)$.

Moreover, similarly to (3.26), define the weight

$$\tilde{W}'(p,\xi)=\begin{bmatrix} I & S(\xi) \\ S^*(\xi) & (N\xi+\Gamma)^{*p}\left\{I-P_+\left(I-\tilde{W}_p\right)P_+\right\}(N\xi+\Gamma)^p \end{bmatrix} \tag{3.35}$$

specifying the Hilbert space

$$L^2_{\mathbb{T}}\left(\tilde{W}'(p,\xi)\right)=\left\{g(\xi)=\begin{pmatrix} v(\xi) \\ u(\xi)\end{pmatrix}:\int_0^{2\pi}\left\langle \tilde{W}'(p,\xi)g(\xi),g(\xi)\right\rangle\frac{d\xi}{2\pi i\xi}<\infty\right\}, \tag{3.34$'$}$$

where $u(\xi)$ and $v(\xi)$ have the same sense as in the definition of space $L^2_{\mathbb{T}}\left(\tilde{W}(p,\xi)\right)$ (3.34).

Specify now the operator functions

$$\tilde{U}_+(1,0)g(\xi)=\overline{\xi}g(\xi);$$

$$\tilde{U}_+(1,p)g(\xi)=\overline{\xi}\begin{bmatrix}\left(\tilde{N}\xi+\tilde{\Gamma}\right)^{*p} & 0 \\ 0 & (N\xi+\Gamma)^{-p}\end{bmatrix}g(\xi), \tag{3.36}$$

where $p\in\mathbb{Z}_+$ and $g(\xi)\in L^2_{\mathbb{T}}\left(\tilde{W}'(p,\xi)\right)$. In this case the model space $\hat{H}_{p,+}$ is given by

$$\tilde{H}_{p,+}=L^2_{\mathbb{T}}\left(\tilde{W}(p,\xi)\right)\ominus\begin{pmatrix} H^2_+\left(\tilde{N}^*,\tilde{\Gamma}^*,\tilde{E}\right) \\ H^2_-\left(N^*,\Gamma^*,E\right)\end{pmatrix}, \tag{3.37}$$

where the Hardy spaces $H^2_-\left(N^*,\Gamma^*,E\right)$ and $H^2_+\left(\tilde{N}^*,\tilde{\Gamma}^*,\tilde{E}\right)$ are obtained from the standard Hardy classes $H^2_-(E)$ and $H^2_+\left(\tilde{E}\right)$ just as in Subsect. V.

Similarly, consider the space

$$\tilde{H}'_{p,+}=L^2_{\mathbb{T}}\left(\tilde{W}'(p,\xi)\right)\ominus\begin{pmatrix}\overline{\xi}\left(\tilde{N}\xi+\tilde{\Gamma}\right)^{*p}H^2_+\left(\tilde{N}^*,\tilde{\Gamma}^*,\tilde{E}\right) \\ \overline{\xi}(N\xi+\Gamma)^{-p}H^2_-\left(N^*,\Gamma^*,E\right)\end{pmatrix}, \tag{3.37$'$}$$

besides the weight $\tilde{W}'(p,\xi)$ is given by (3.35). Specify the operator

$$\tilde{R}_{p,+}=P_{\tilde{H}'_{p,+}}\begin{bmatrix}\xi\left(\tilde{N}^*\overline{\xi}+\tilde{\Gamma}^*\right)^{-p} & 0 \\ 0 & \xi(N\xi+\Gamma)^p\end{bmatrix}P_{\tilde{H}_{p,+}}, \tag{3.38}$$

where $P_{\tilde{H}_{p,+}}$ and $P_{\tilde{H}'_{p,+}}$ are the corresponding orthoprojectors on $\tilde{H}_{p,+}$ (3.37) and $\tilde{H}'_{p,+}$ (3.37$'$). It is clear that the operators T_1^* and $T^*(1,p)=T_1^*T_2^{*p}$ in space $\tilde{H}_{p,+}$ are given by

$$\left(\tilde{T}_1^*f\right)(\xi)=P_{\tilde{H}_{p,+}}\overline{\xi}f(\xi);$$

$$\left(\tilde{T}^*(1,p)f\right)(\xi) = P_{\tilde{H}_{p,+}}\overline{\xi}\begin{bmatrix} \left(\tilde{N}\overline{\xi}+\tilde{\Gamma}^*\right)^p & 0 \\ 0 & (N\xi+\Gamma)^{-p} \end{bmatrix}\left(\tilde{R}_{p,+}f\right)(\xi) \qquad (3.39)$$

for all $f(\xi) \in \tilde{H}_{p,+}$, where $P_{\tilde{H}_{p,+}}$ is the orthoprojector on $\tilde{H}_{p,+}$, and $\tilde{R}_{p,+}$ is given by (3.38). From this it easily follows that the initial operator system $\{T_1^*, T_2^*\}$, defined in H, in space $\tilde{H}_{1,+}$ (3.37) is given by

$$\left(\tilde{T}_1^* f\right)(\xi) = P_{\tilde{H}_{1,+}}\overline{\xi} f(\xi);$$

$$\left(\tilde{T}_2^* f\right)(\xi) = P_{\tilde{H}_{1,+}}\begin{bmatrix} \tilde{N}^*\overline{\xi}+\tilde{\Gamma}^* & 0 \\ 0 & (N\xi+\Gamma)^{-1} \end{bmatrix}\left(\tilde{R}_{1,+}f\right)(\xi), \qquad (3.40)$$

where $f(\xi) \in \tilde{H}_{1,+}$.

Theorem 4. *Let* V_s, $\overset{+}{V_s}$ *(3.1) be the simple [2, 3] commutative unitary extensions of a commutative operator system* $\{T_1, T_2\}$ *from the class* $C(T_1)$ *(1.3), besides the suppositions of Lem. 1.1 are met and* $\dim E = \dim \tilde{E} < \infty$. *Then the isometric dilation* $\overset{+}{U}(1,p)$ *(1.24), given in the Hilbert space* $\mathcal{H}_{N^*,\Gamma^*}$ *(3.22), is unitary equivalent to the functional model:* $\tilde{U}_+(1,0)$ *(3.36), when* $p = 0$ *in* $L^2_{\mathbb{T}}\left(\tilde{W}(p,\xi)\right)$ *(3.34), and to the operator* $\tilde{U}_+(1,p)$ *(3.36) mapping the space* $L^2_{\mathbb{T}}\left(\tilde{W}'(p,\xi)\right)$ *(3.34) in* $L^2_{\mathbb{T}}\left(\tilde{W}(p,\xi)\right)$ *(3.34). Moreover, the operators* T_1^* *and* $T^*(1,p) = T_1^* T_2^{*p}$ *(1.20) given in* H *are unitary equivalent to the functional model* $\tilde{T}_1^*$ *(3.40) in* $\tilde{H}_{p,+}$ *(3.37) for all* $p \in \mathbb{Z}_+$ *and to the operator* $\tilde{T}_1^*(1,p)$ *(3.39) only in the concrete model space* $\tilde{H}_{p,+}$ *(3.37) when* $p \in \mathbb{N}$.

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