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# On Subharmonic Functions of the First Order with Restrictions on the Real Axis

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Subharmonic functions $v$ of the first proximate order $\rho(r)$ with the integral $\int\limits_0^R \frac{t^{1-\rho(t)}(v(t)+v(-t))}{1+t^2}dt$ bounded with respect to $R$ are studied. This is an extension of a result by N.I. Akhiezer, who studied the case $\rho(r) \equiv 1$, $v(z) = \ln|f(z)|$, where $f(z)$ is an entire function.

*Key words*: proximate order, functions of the class A, functions of completely regular growth.

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According to the definition given by N.N. Meiman [1], a function $f$ is of the class $A$ if

$$\sum_{k=1}^{\infty} \left| \Im \frac{1}{a_k} \right| < \infty, \tag{1}$$

where $a_k$ are zeros of $f$. Functions of the class $A$ play an important role in the theory of entire functions. In applications they are most often of exponential type, that is, of the first order and normal type.

In [2] N.I. Akhiezer proved that an entire function of exponential type belongs to the class $A$ if the integral

$$\int\limits_0^R \frac{\ln|F(x)F(-x)|}{1+x^2}dx \tag{2}$$

is bounded with respect to $R$.

The Cartwright class of entire functions of exponential type with the bounded integral

$$\int\limits_{-\infty}^{\infty} \frac{\ln_+|F(t)|}{1+t^2}dt \tag{3}$$

is also well known. From the results of [3] it follows that the functions of the Cartwright class are of the class $A$ and of completely regular growth.

A chapter of book [4] and a section of review [5] are devoted to the functions of the class $A$. The integrals of form (3) were also studied in the book by P. Koosis [6]. In the present paper the concepts of proximate and formal orders are widely used.

A differentiable function $\rho(r)$ on the half-axis $(0,\infty)$ is said to be of proximate order in the sense of Valiron if it satisfies the following two conditions:

1) $\lim\limits_{r\to+\infty} \rho(r) = \rho \in (-\infty,\infty)$,

2) $\lim\limits_{r\to+\infty} r\ln r\rho'(r) = 0$.

Properties of the proximate order are presented, e.g., in [4]. Some more results can be found in [7].

The function $r^{\rho(r)}$ is denoted by $V(r)$.

The proximate order $\rho(r)$ is said to be a formal order of a function $v$ subharmonic in the plane $\mathbb{C}$ (in the upper half-plane $\mathbb{C}^+$) if there exists a constant $M_1$ such that

$$v(re^{i\theta}) \le M_1 V(r), \quad r \ge 1$$

for $\theta \in [0,2\pi]$ $(\theta \in (0,\pi))$.

A class of functions of formal order $\rho(r)$ is denoted by $SF(\rho(r))$.

We say that a measure $\mu$ has a formal order $\rho(r)$ if there exists a constant $M_2$ such that for all sufficiently large $r$

$$\mu(B(0,r)) \le M_2 V(r), \qquad B(0,r) = \{z : |z| \le r\}.$$

A class of measures of formal order $\rho(r)$ is denoted by $\mathfrak{M}(\rho(r))$.

We should note that the Riesz measure $\mu$ of subharmonic function $v(z)$ of the formal order $\rho(r)$ is also of the formal order $\rho(r)$.

The definitions above are in [8].

For the subharmonic functions $v$ of formal order $\rho(r)$, $\rho(r) \to 1$ as $r \to \infty$, the following two integrals

$$\int\limits_0^R \frac{W(t)}{1+t^2}(v(t)+v(-t))dt \tag{4}$$

and

$$\iint\limits_{B(0,\,R)} \int\limits_0^\infty \frac{W(t)}{1+t^2} \ln\left|1-\frac{t^2}{\zeta^2}\right| dt d\mu(\zeta), \tag{5}$$

are considered, where $W(t) = t^{1-\rho(t)}$, $\mu$ is the Riesz measure of the function $v(z)$. The first integral is analogical to (2), and the second one to the partial sum of the series from (1).

The proximate order $\rho(r)$ was introduced to obtain the estimates of functions in a neighborhood of infinity. Therefore, the behavior of $\rho(r)$, as $r \to 0$, is not significant. In addition, the formulations and proves of many theorems are simplified if we consider the integrals of the form

$$\int_0^a V(t)K(x,t)dt.$$

For example, integral (4) arises in the presented paper. Without any restrictions on the behavior of $\rho(r)$ in a neighborhood of zero, the integral (4), generally speaking, is divergent. And in the concrete formulations we ought to replace the low limit of integration by 1.

However, sometimes the appropriate restrictions on the behavior of $\rho(r)$ in a neighborhood of zero are more preferable. In our case the condition $W(r)W\left(\frac{1}{r}\right)$ $= 1$ turns out to be convenient.

In Theorem 1 it is proved that the integrals (4) and (5) are bounded, or are not bounded simultaneously. This is the main result of the paper, the remaining statements are its corollaries. In particular, under some additional restrictions on the function $v(z)$ we obtain that (5) is bounded if and only if the following two integrals are convergent:

$$\iint\limits_{\mathbb{C}\setminus B(0,\, r_0)} \frac{W(\tau)}{\tau} |\sin\theta|\, d\mu(\zeta) \tag{6}$$

and

$$\iint\limits_{\mathbb{C}\setminus B(0,\, r_0)} W'(\tau)\, d\mu(\zeta), \tag{7}$$

where $\zeta = \tau e^{i\theta}$, and $r_0$ is an arbitrary fixed strictly positive number.

If $\rho(r) \equiv 1$, then the second integral vanishes. In this particular case Th. 1 transforms into the extension of the Akhiezer theorem for the class of subharmonic functions. Thus, one can say that Th. 1 generalizes the Akhiezer theorem to the case of subharmonic functions of a proximate order $\rho(r)$, $\rho(r) \to 1$ as $r \to +\infty$.

We should point out that the convergence conditions for (6) and (7) are independent. Hence, the case of a general proximate order $\rho(r)$ and the case $\rho(r) \equiv 1$ are significantly different.

Note that many statements on entire or subharmonic functions of order $\rho$ can be easily extended to the functions of order $\rho(r)$. However, Akhiezer's proof can not be extended to the case of general proximate order.

Now we give some notation for the sets used in the paper:

$$B(z_0, R) = \{z : |z - z_0| \le R\},$$

$$K(R_1, R_2) = \{z : R_1 \le |z| < R_2\}.$$

Further we will formulate the definition of the limit set of measure according to V.S. Azarin [10].

Consider the transformation $(\cdot)_t$ of measure $\mu \in \mathfrak{M}(\rho(r))$, which is defined by the equality

$$\mu_t(E) = \frac{\mu(tE)}{V(t)}$$

for any Borel set $E$.

The Azarin limit set $Fr[\mu]$ of a measure $\mu \in \mathfrak{M}(\rho(r))$ is defined as the set of measures given by the condition

$$\nu = \lim_{n\to\infty} \mu_{t_n} \quad \textit{for some sequence } (t_n), \quad t_n \to +\infty.$$

Here the limit is taken in the sense of distributions. This means that for a test function $\varphi(z)$ (that is, compactly supported and infinitely differentiable) the following equality holds:

$$\lim_{n\to\infty} \iint \varphi(z) d\mu_{t_n}(z) = \iint \varphi(z) d\nu(z).$$

A definition of the normal set is also needed.

A set $E$ is said to be normal for a measure $\mu$ if $\mu(\partial E) = 0$.

Let $v(z)$ be a subharmonic function and $\mu$ be its Riesz measure,

$$v_1(z) = v(z) - \iint\limits_{B(0,1)} \ln|z - \zeta| d\mu(\zeta), \qquad v_2(z) = v_1(z) - v_1(0).$$

Note that the function $v_1(z)$ is harmonic in the disk $|z| < 1$. Also, let $\mu_1$ be a restriction of measure $\mu$ on the exterior of an open unit disk. The measure $\mu_1$ is the Riesz measure of subharmonic function $v_2$.

Integral (4) for the function $v_2$ is bounded together with the same integral for the function $v$. Moreover, the Azarin limit sets of measures $\mu$ and $\mu_1$ coincide.

Hence, further we may assume that the function $v(z)$ is harmonic in the disk $|z| < 1$ and $v(0) = 0$, unless the contrary is stated.

**Lemma 1.** *The limit set of the integral*

$$I_1(R) = \iint\limits_{K(R,\infty)} \int\limits_0^R \frac{W(t)}{t^2} \ln\left|1 - \frac{t^2}{\zeta^2}\right| dt d\mu(\zeta)$$

*in the direction $R \to +\infty$ is of the form*

$$\left\{ \iint\limits_{K(1,\,\infty)} \int\limits_0^1 \frac{1}{u^2} \ln\left|1 - \frac{u^2}{{\zeta_1}^2}\right| du d\nu(\zeta_1), \quad \nu \in E^1_\mu \right\}. \tag{8}$$

*Here $\mu$ is the Riesz measure of subharmonic function of the class $SF(\rho(r))$, $E^1_\mu$ is the set of measures $\nu$ which coincides with the limit set in the direction $t \to \infty$ of the family of measures $\mu^1_t$, where $\mu^1_t$ is the restriction of the measure $\mu_t$ to the set $\{z : |z| \geq 1\}$.*

P r o o f. First of all, let us give some helpful inequalities.

From the Poisson–Jensen formula for a subharmonic function $v$ under the assumption $v(0) = 0$ it follows that there exists a constant $M_3$ such that

$$\int\limits_0^R \frac{\mu(t)}{t} dt \leq M_3 V(R). \tag{9}$$

From (9) in a standard way (see, for example, [4, Ch. 1, §5]) we get

$$\mu(R) \leq M_4 V(R). \tag{10}$$

Let us formulate another useful for us statement ([7, Th. 4]).

If $\rho(r)$ is an arbitrary zero proximate order for which $V(r)V\left(\frac{1}{r}\right) = 1$, then there exists a zero proximate order $\rho_1(r)$ such that

$$V(xt) \leq V(x)V_1(t), \quad x > 0,\ t > 0, \tag{11}$$

where $V_1(t) = t^{\rho_1(t)}$.

As it was said above, we assume that

$$W(r)W\left(\frac{1}{r}\right) = 1.$$

It allows us to apply (11) to the function $W(r)$.

Then consider the integral $I_1(R)$. Replacing $\zeta$ by $R\zeta_1$ in the exterior integral and $t$ by $uR$ in the inner integral, we obtain

$$I_1(R) = \iint\limits_{K(1,\infty)} \int\limits_0^1 \frac{W(uR)}{W(R)} \frac{1}{u^2} \ln\left|1 - \frac{u^2}{{\zeta_1}^2}\right| du d\mu_R(\zeta_1)$$

$$= \iint\limits_{\mathbb{C}} \int\limits_0^1 \frac{W(uR)}{W(R)} \frac{1}{u^2} \ln\left|1 - \frac{u^2}{{\zeta_1}^2}\right| du d\mu^1_R(\zeta_1).$$

Choose a sequence $(R_n)$, $R_n \to +\infty$, such that its limit $(I_1(R_n))$ exists (it may be improper limit).

Since the sequence $\left(\mu^1_{R_n}\right)$ is bounded, then we may additionally assume ([10, Th. 1.2.1]) that it converges as the sequence of distributions to a measure $\nu$. Note that $\nu \in E^1_\mu$.

Represent the integral $I_1(R_n)$ in the form of the following sum:

$$I_1(R_n) = \iint\limits_{B(0,N)} \int\limits_0^1 \frac{W(uR_n)}{W(R_n)} \frac{1}{u^2} \ln\left|1 - \frac{u^2}{{\zeta_1}^2}\right| dud\mu^1_{R_n}(\zeta_1)$$

$$+ \iint\limits_{K(N,\infty)} \int\limits_0^1 \frac{W(uR_n)}{W(R_n)} \frac{1}{u^2} \ln\left|1 - \frac{u^2}{{\zeta_1}^2}\right| dud\mu^1_{R_n}(\zeta_1) = I^1_1(R_n) + I^2_1(R_n). \tag{12}$$

Here $N$ is chosen so that the support of measure $\nu$ does not contain the circle $|z| = N$.

First we prove that

$$\lim_{n\to\infty} I^1_1(R_n) = \iint\limits_{B(0,N)} \int\limits_0^1 \frac{1}{u^2} \ln\left|1 - \frac{u^2}{{\zeta_1}^2}\right| dud\nu(\zeta_1). \tag{13}$$

To end this we take an arbitrary $\delta \in (0, 1)$ and represent the integral $I^1_1(R_n)$ in the form of the following sum:

$$I^1_1(R_n) = \iint\limits_{B(0,N)} \int\limits_0^\delta \frac{W(uR_n)}{W(R_n)} \frac{1}{u^2} \ln\left|1 - \frac{u^2}{{\zeta_1}^2}\right| dud\mu^1_{R_n}(\zeta_1)$$

$$+ \iint\limits_{B(0,N)} \int\limits_\delta^1 \frac{W(uR_n)}{W(R_n)} \frac{1}{u^2} \ln\left|1 - \frac{u^2}{{\zeta_1}^2}\right| dud\mu^1_{R_n}(\zeta_1) = I^{1,1}_1(n,\delta) + I^{1,2}_1(n,\delta).$$

Let us consider the second summand in the right-hand part of the equality. Since the disk $B(0,N)$ is normal with respect to measure $\nu$, then Th. $0.5'$ [9] yields

$$\lim_{n\to\infty} I^{1,2}_1(n,\delta) = \iint\limits_{B(0,N)} \int\limits_\delta^1 \frac{1}{u^2} \ln\left|1 - \frac{u^2}{{\zeta_1}^2}\right| dud\nu(\zeta_1).$$

To prove this equality we have to apply the following property of proximate order ([4, Ch. 1, § 12]):

$$\frac{W(uR)}{W(R)} \rightrightarrows 1, \quad R \to +\infty, \quad u \in [a,b],\ 0 < a < b < +\infty. \tag{14}$$

Using (11) for $I_1^{1,1}(n,\delta)$, we get the estimates

$$\left|I_1^{1,1}(n,\delta)\right| \le \iint\limits_{B(0,N)} \int\limits_0^\delta \frac{V_1(u)}{u^2}\left|\ln\left|1-\frac{u^2}{{\zeta_1}^2}\right|\right| du d\mu_{R_n}^1(\zeta_1)$$

$$\le M_5 \int\limits_0^\delta V_1(u)du \iint\limits_{B(0,N)} \frac{d\mu_{R_n}^1(\zeta_1)}{|\zeta_1|^2} \le M_6 \int\limits_0^\delta V_1(u)du.$$

From this it follows that $I_1^{1,1}(n,\delta)$ tends to zero uniformly with respect to $n$ as $\delta \to +0$. Together with the equality

$$\lim_{\delta\to 0} \iint\limits_{B(0,N)} \int\limits_0^\delta \frac{1}{u^2}\ln\left|1-\frac{u^2}{{\zeta_1}^2}\right| du d\nu_1(\zeta_1) = 0$$

it gives the demanded relation (13).

Now we estimate the summand $I_1^2(R_n)$.

$$\left|I_1^2(R_n)\right| \le \iint\limits_{K(N,\infty)} \int\limits_0^1 \frac{W(uR_n)}{W(R_n)} \frac{2}{{\tau_1}^2} du d\mu_{R_n}^1(\zeta_1)$$

$$\le 2 \iint\limits_{K(N,\infty)} \int\limits_0^1 V_1(u)du \frac{d\mu_{R_n}}{\tau_1^2} \le M_7 \int\limits_N^\infty \frac{d\mu_{R_n}^1(\tau_1)}{{\tau_1}^2}, \qquad \tau_1 = |\zeta_1|.$$

Integrating by parts and using (10) and (11) for $\alpha \in (0,1)$ and $N > N(\alpha)$, we obtain

$$\int\limits_N^\infty \frac{d\mu_R(\tau_1)}{{\tau_1}^2} = \left.\frac{\mu_R(\tau_1)}{{\tau_1}^2}\right|_N^\infty + 2\int\limits_N^\infty \frac{\mu_R(\tau_1)}{{\tau_1}^3} d\tau_1 \le 2\int\limits_N^\infty \frac{\mu_R(\tau_1)}{{\tau_1}^3} d\tau_1$$

$$= 2\int\limits_N^\infty \frac{1}{V(R)} \frac{\mu(R\tau_1)}{{\tau_1}^3} d\tau_1 \le 2M_4 \int\limits_N^\infty \frac{V(R\tau_1)}{V(R)} \frac{d\tau_1}{{\tau_1}^3} \le 2M_4 \int\limits_N^\infty \frac{d\tau_1}{{\tau_1}^{2-\alpha}} = \frac{M_8}{N^{1-\alpha}}.$$

Thus, for $I_1^2(R_n)$ we have

$$\left|I_1^2(R_n)\right| = \left|\iint\limits_{K(N,\infty)} \int\limits_0^1 \frac{W(uR_n)}{W(R_n)} \frac{1}{u^2} \ln\left|1-\frac{u^2}{{\zeta_1}^2}\right| du d\mu_{R_n}(\zeta_1)\right| \le \frac{M_9}{N^{1-\alpha}}.$$

The following estimate also holds:

$$\left|\iint\limits_{K(N,\infty)}\int\limits_0^1 \ln\left|1-\frac{u^2}{\zeta_1^2}\right| d\nu(\zeta)\right| \le \frac{M_{10}}{N^{1-\alpha}}.$$

Taking into account (12), we get

$$\left|I_1(R_n) - \iint\limits_{\mathbb{C}}\int\limits_0^1 \frac{1}{u^2}\ln\left|1-\frac{u^2}{{\zeta_1}^2}\right| dud\nu(\zeta_1)\right|$$

$$\le \left|I_1(R_n) - \iint\limits_{B(0,N)}\int\limits_0^1 \frac{1}{u^2}\ln\left|1-\frac{u^2}{{\zeta_1}^2}\right| dud\nu(\zeta_1)\right| + \frac{M_{11}}{N^{1-\alpha}}.$$

Passing to the limit (it means the upper limit) first, when $n \to \infty$, and then, when $N \to \infty$, we obtain that the limit set of integral $I_1(R)$ in the direction $R \to +\infty$ consists of the values given by formula (8).

Conversely, if we take the sequence $(R_n)$ such that the sequence $(\mu^1_{R_n})$ converges to the measure $\nu$ in the sense of distributions, then we find that the right-hand part of formula (8) is held in the limit set of integral $I_1(R)$ in the direction $R \to +\infty$. The lemma is proved.

R e m a r k 1. Since for a function $\nu_1 \in E^1_\mu$ the inequality $\nu_1(B(0,R)) \le MR$ holds for all $R > 0$ ([10, Th. 1.2.2]), then the limit set of integral $I_1(R)$ in the direction $R \to +\infty$ is bounded, and hence, the value $I_1(R)$ is bounded on the half-axis $[1,\infty)$.

**Lemma 2.** *The limit set of the integral*

$$I_2(R) = \iint\limits_{B(0,R)}\int\limits_R^\infty \frac{W(t)}{t^2}\ln\left|1-\frac{t^2}{\zeta^2}\right| dtd\mu(\zeta)$$

*in the direction $R \to +\infty$ is of the form*

$$\left\{\iint\limits_{B(0,1)}\int\limits_1^\infty \frac{1}{u^2}\ln\left|1-\frac{u^2}{{\zeta_1}^2}\right| dud\nu(\zeta_1), \quad \nu \in E^2_\mu\right\}.$$

*Here $\mu$ is the Riesz measure of subharmonic function of the class $SF(\rho(r))$, $E^2_\mu$ is the set of measures $\nu$, which coincides with the limit set in the direction $t \to +\infty$ of the family of measures $\mu^2_t$, where $\mu^2_t$ is the restriction of measure $\mu_t$ to the set $\{\zeta : |\zeta| \le 1\}$.*

The proof of Lem. 2 is not presented here, as it is completely analogous to that of Lem. 1.

R e m a r k 2. The value $I_2(R)$ is bounded.

**Theorem 1.** *Let $v(z) \in SF(\rho(r))$, $\rho(r) \to 1$ as $r \to +\infty$, in the plane $\mathbb{C}$. Then*

$$\int_0^R \frac{W(t)}{1+t^2}(v(t)+v(-t))dt = \iint_{B(0,R)} \int_0^\infty \frac{W(t)}{1+t^2} \ln\left|1-\frac{t^2}{\zeta^2}\right| dt d\mu(\zeta) + \varphi(R), \tag{15}$$

*where $\mu(\zeta)$ is the Riesz measure of function $v$, $\varphi(R)$ is the function bounded on $(0,\infty)$.*

P r o o f. As it was pointed out above, in the proof we may assume that the support of measure $\mu$ does not contain the set $B(0,1)$ and $v(0)=0$. In this case in the formulation of the theorem the integrals (1) and (2) can be replaced by the integrals

$$\int_0^R \frac{W(t)}{t^2}(v(t)+v(-t))dt \tag{16}$$

and

$$\iint_{B(0,R)} \int_0^\infty \frac{W(t)}{t^2} \ln\left|1-\frac{t^2}{\zeta^2}\right| dt d\mu(\zeta). \tag{17}$$

An analogical statement was proved by Levin ([4, Ch. 5, § 1]). In our case his proof is valid.

By the Brelot–Hadamar theorem ([11, Ch. 4, § 2]) for the function $v$ (under our additional assumptions) the formula

$$v(z) = \iint_{\mathbb{C}} \Re\left\{\ln\left(1-\frac{z}{\zeta}\right)+\frac{z}{\zeta}\right\} d\mu(\zeta) + c\,\Im z \tag{18}$$

is true with some real constant $c$. From this it follows that

$$\int_0^R \frac{W(t)}{t^2}(v(t)+v(-t))dt = \int_0^R \int_{\mathbb{C}} \frac{W(t)}{t^2} \ln\left|1-\frac{t^2}{\zeta^2}\right| d\mu(\zeta) dt.$$

Let us interchange the order of integration in the last integral. The function $\int_0^R \frac{W(t)}{t^2} \ln\left|1-\frac{t^2}{\zeta^2}\right| dt$ is continuous in the variable $\zeta$ on the set $\mathbb{C}\setminus\{0\}$ and of the form $O\left(\frac{1}{|\zeta|^2}\right)$ as $\zeta \to \infty$. Hence, the integrand from the integral in the right-hand part of the last equality falls into the $L_1$-space by measure $dt \times d\mu$. Due to the Fubini theorem we have

$$\int_0^R \frac{W(t)}{t^2}(v(t)+v(-t))dt = \iint_{\mathbb{C}} \int_0^R \frac{W(t)}{t^2} \ln\left|1-\frac{t^2}{\zeta^2}\right| dt d\mu(\zeta). \tag{19}$$

Now we represent the last integral in this equality as the following sum:

$$\iint_{\mathbb{C}} \int_0^R \frac{W(t)}{t^2} \ln\left|1-\frac{t^2}{\zeta^2}\right| dt d\mu(\zeta) = \iint_{B(0,\,R)} \int_0^\infty \frac{W(t)}{t^2} \ln\left|1-\frac{t^2}{\zeta^2}\right| dt d\mu(\zeta)$$

$$+ \iint_{K(R,\infty)} \int_0^R \frac{W(t)}{t^2} \ln\left|1-\frac{t^2}{\zeta^2}\right| dt d\mu(\zeta) - \iint_{B(0,\,R)} \int_R^\infty \frac{W(t)}{t^2} \ln\left|1-\frac{t^2}{\zeta^2}\right| dt d\mu(\zeta).$$

Then from (19) we get

$$\int_0^R \frac{W(t)}{t^2}\,(v(t)+v(-t))\,dt = \iint_{B(0,\,R)} \int_0^\infty \frac{W(t)}{t^2} \ln\left|1-\frac{t^2}{\zeta^2}\right| dt d\mu(\zeta) + I_1(R) - I_2(R), \tag{20}$$

where $I_1(R)$, $I_2(R)$ are the integrals defined in Lems. 1 and 2, respectively. From Remarks 1 and 2 to the corresponding lemmas one can see that the value $\varphi(R) = I_1(R) - I_2(R)$ is bounded. Together with (20) it gives us the assertion of the theorem.

Let us notice that if $\rho(r) \equiv 1$, then we obtain the Akhiezer theorem ([2], p.475).

Further there will be given a corollary of Th. 1.

**Theorem 2.** *Let $v(z)$ be a subharmonic function of completely regular growth with respect to the proximate order $\rho(r)$, $\rho(r) \to 1$ as $r \to +\infty$, in $\mathbb{C}$. Then the convergence of the integral*

$$\int_0^\infty \frac{W(t)}{1+t^2}(v(t)+v(-t))dt \tag{21}$$

*is equivalent to the convergence of the integral*

$$\iint_{\mathbb{C}} \int_0^\infty \frac{W(t)}{1+t^2} \ln\left|1-\frac{t^2}{\zeta^2}\right| dt d\mu(\zeta). \tag{22}$$

P r o o f. Since the function $v$ is of completely regular growth, then the limit set of its Riesz measure is single-element ([10, Th. 1.3.1]). Moreover, the support of the limit measure does not contain any circle. According to Th. 0.5′ [9] in this case $E^1_\mu$ and $E^2_\mu$ are single-element sets, and hence, each of the two values $I_1(R)$ and $I_2(R)$ has its limit as $R \to +\infty$. Now equality (20) implies the assertion of the theorem.

It turns out that under some additional restrictions on the function $\rho(r)$ the boundness of (4) and (5) implies the existence of the limits of these integrals as $R \to +\infty$.

**Theorem 3.** *Let $v(z) \in SF(\rho(r))$, $\rho(r) \to 1$ as $r \to +\infty$, in $\mathbb{C}$. Also let the following two conditions be valid:*

*1) $\eta(r) = \rho(r) - 1 + r\rho'(r) \ln r \le 0$,*

*2) $rW'(r) = W_1(r)$, where $W_1(r) = r^{\rho_1(r)}$, $\rho_1(r)$ is some zero proximate order.*

*Then the integral*

$$\int_0^R \frac{W(t)}{1+t^2}(v(t)+v(-t))dt \tag{23}$$

*is bounded from above if and only if the following two integrals,*

$$\iint_{\mathbb{C}} \frac{W(\tau)}{\tau} |\sin\theta| d\mu(\zeta) \tag{24}$$

*and*

$$\iint_{\mathbb{C}} \frac{W_1(\tau)}{\tau} d\mu(\zeta), \tag{25}$$

*are convergent. Here $\mu$ is the restriction of the Riesz measure of the function $v$ to the ring $K(1,\infty)$.*

P r o o f. As it was said above, without loss of generality, we assume that the function $v$ is harmonic in the unit disk and $v(0) = 0$. In this case instead of (23) we may consider the integral

$$\int_0^R \frac{W(t)}{t^2}(v(t)+v(-t))dt. \tag{26}$$

Denote

$$w(\zeta) = \int_0^\infty \frac{W(t)}{t^2} \ln\left|1 - \frac{t^2}{\zeta^2}\right| dt. \tag{27}$$

The function $w(\zeta)$ satisfies the following asymptotic formula:

$$w(\zeta) = |\sin\theta| \frac{W(\tau)}{\tau} + (1+o(1))H(\theta)\frac{W_1(\tau)}{\tau}, \tag{28}$$

where

$$H(\theta) = \int_0^\infty \left( \frac{1}{u^2} \ln \left| 1 - u^2 e^{2i\theta} \right| - \frac{\cos\theta}{u} \ln \left| \frac{u - e^{i\theta}}{u + e^{i\theta}} \right| \right) du.$$

Now let us evaluate $H(\theta)$. Integrating by parts, for the first summand of the integrand in the last integral we obtain

$$\int_0^\infty \frac{1}{u^2} \ln \left| 1 - u^2 e^{2i\theta} \right| du = 2\Re \int_0^\infty \frac{du}{u^2 - e^{-2i\theta}} = \Re \int_{-\infty}^\infty \frac{du}{u^2 - e^{-2i\theta}}.$$

The evaluation by the residue theorem gives us

$$\int_0^\infty \frac{1}{u^2} \ln \left| 1 - u^2 e^{2i\theta} \right| du = \pi |sin\theta|.$$

Integrating by parts again, for the second summand in the integrand we find

$$\int_0^\infty \frac{\cos\theta}{u} \ln \left| \frac{u - e^{i\theta}}{u + e^{i\theta}} \right| du = -2\cos\theta\, \Re \int_0^\infty \frac{e^{i\theta} \ln u}{u^2 - e^{2i\theta}} du.$$

Applying again the residue theorem, we get

$$\int_0^\infty \frac{\cos\theta}{u} \ln \left| \frac{u - e^{i\theta}}{u + e^{i\theta}} \right| du = -\pi \left( \frac{\pi}{2} - \theta \right) \cos\theta, \quad \theta \in (0, \pi).$$

From the evenness of the integral in the variable $\theta$ it follows that for $\theta \in (-\pi, 0)$

$$\int_0^\infty \frac{\cos\theta}{u} \ln \left| \frac{u - e^{i\theta}}{u + e^{i\theta}} \right| du = -\pi \left( \frac{\pi}{2} + \theta \right) \cos\theta.$$

And, finally, using the continuity of $H(\theta)$ on the segment $[-\pi, \pi]$, we arrive at the formula

$$H(\theta) = \begin{cases} \pi \left( \frac{\pi}{2} + \theta \right) \cos\theta - \pi \sin\theta, & \theta \in [-\pi, 0], \\ \pi \left( \frac{\pi}{2} - \theta \right) \cos\theta + \pi \sin\theta, & \theta \in [0, \pi]. \end{cases}$$

It is easy to see that

$$\pi \le H(\theta) \le \frac{\pi^2}{2}.$$

Taking into account these inequalities, as well as the relation

$$W_1(r) = rW'(r) = -W(r)\eta(r) \geq 0,$$

from formula (20) and representation (28) the assertion of the theorem follows immediately.

It remains to prove formula (28). From (27), integrating by parts we get

$$w(\zeta) = -\int_0^\infty W(t)d\psi(t,\zeta) = \lim_{\varepsilon\to+0}\left(-W(t)\psi(t,\zeta)|_\varepsilon^\infty + \int_\varepsilon^\infty W'(t)\psi(t,\zeta)dt\right),$$

where

$$\psi(t,\zeta) = \int_t^\infty \frac{1}{x^2}\ln\left|1-\frac{x^2}{\zeta^2}\right|dx.$$

It is easy to verify that for $t>0$ and $\theta\in(0,\pi)$

$$\psi(t,\zeta) = \Re\left\{\frac{1}{t}\ln\left(1-\frac{t^2}{\zeta^2}\right) - \frac{1}{\zeta}\ln\left(\frac{t-\zeta}{t+\zeta}\right)\right\}$$

$$= \frac{1}{t}\ln\left|1-\frac{t^2}{\zeta^2}\right| - \frac{\cos\theta}{\tau}\ln\left|\frac{t-\zeta}{t+\zeta}\right| + \frac{\sin\theta}{\tau}\operatorname{arctg}\frac{2t\tau\sin\theta}{t^2-\tau^2} + \frac{\pi\sin\theta}{\tau}\chi_{[0,\tau)}(t).$$

To get this one should use the following relations:

$$\Im\ln\left(\frac{t-\zeta}{t+\zeta}\right) = \Im\ln(t-\zeta)(t+\overline{\zeta}) = \Im\ln(t^2-\tau^2-2i\tau t\sin\theta)$$

$$= -\operatorname{arctg}\frac{2t\tau\sin\theta}{t^2-\tau^2} + k(t)\pi, \qquad \lim_{t\to+\infty}\Im\ln\left(\frac{t-\zeta}{t+\zeta}\right) = 0$$

and the continuity of function $\psi$ in the variable $t$ on the half-axis $(0,\infty)$.

Thus, we obtain

$$w(\zeta) = \pi\sin\theta\frac{W(\tau)}{\tau} + \frac{\sin\theta}{\tau}\int_0^\infty \frac{W_1(t)}{t}\operatorname{arctg}\frac{2t\tau\sin\theta}{t^2-\tau^2}dt$$

$$+ \int_0^\infty W_1(t)\left(\frac{1}{t^2}\ln\left|1-\frac{t^2}{\zeta^2}\right| - \frac{\cos\theta}{t\tau}\ln\left|\frac{t-\zeta}{t+\zeta}\right|\right)dt. \qquad (29)$$

Consider the second summand in the right-hand part of (29). We denote this integral by $I_5(\tau)$. Changing the variable $t = u\tau$ in it, we have

$$I_5(\tau) = \sin\theta\frac{W_1(\tau)}{\tau}\int_0^\infty \frac{W_1(u\tau)}{W_1(\tau)}\frac{1}{u}\operatorname{arctg}\frac{2u\sin\theta}{u^2-1}du.$$

After having changed the variable $u = \frac{1}{v}$ one can make sure that

$$\int\limits_0^\infty \frac{1}{u} \operatorname{arctg} \frac{2u \sin\theta}{u^2 - 1} du = 0.$$

Using the last equality, as well as the mentioned above property of a proximate order (14) and inequality (11), and reasoning as in the proof of Lem. 1, we arrive at the following asymptotic formula:

$$I_5(\tau) = o(1) \frac{W_1(\tau)}{\tau}, \quad \tau \to \infty.$$

Then consider the last summand in the right-hand part of (29). Denote this integral by $I_6(\tau)$ and change the variable $t = u\tau$ in it. We obtain

$$I_6(\tau) = \frac{W_1(\tau)}{\tau} \int\limits_0^\infty \frac{W_1(u\tau)}{W_1(\tau)} \left( \frac{1}{u^2} \ln \left| 1 - u^2 e^{2i\theta} \right| - \frac{\cos\theta}{u} \ln \left| \frac{u - e^{i\theta}}{u + e^{i\theta}} \right| \right) du.$$

From here we derive the following asymptotic formula:

$$I_6(\tau) = \frac{W_1(\tau)}{\tau} \left( H(\theta) + o(1) \right), \qquad \tau \to \infty.$$

And now, due to the evenness of function $w(\tau e^{i\theta})$ in the variable $\theta$ from (29) there follows the demanded representation. The theorem is proved.

**Theorem 4.** *Let $v(z) \in SF(\rho(r))$, $\rho(r) \to 1$ as $r \to \infty$, in the plane $\mathbb{C}$ and $W_1(r) = rW'(r)$.*

*Then the convergence of integrals (24) and (25) implies the boundness of integral (23).*

P r o o f. The assertion of the theorem follows immediately from (28) and (20).

If we additionally assume that $v(z)$ is the function of completely regular growth, then using Th. 2 one can get the following corollaries of Ths. 3 and 4.

**Theorem 5.** *Let $v(z)$ be a subharmonic function of completely regular growth with respect to the proximate order $\rho(r)$, $\rho(r) \to 1$ as $r \to +\infty$, in the plane $\mathbb{C}$. Also let the following two conditions be valid:*

*1) $\eta(r) = \rho(r) - 1 + r\rho'(r) \ln r \le 0$,*

*2) $\rho(r)$ is such that $rW'(r) = W_1(r)$, where $W_1(r) = r^{\rho_1(r)}$, $\rho_1(r)$ is some zero proximate order.*

*Then the convergence of integral (23) is equivalent to the convergence of integrals (24) and (25).*

P r o o f. The assertion of the theorem follows immediately from (28) and Th. 2.

**Theorem 6.** *Let $v(z)$ be a subharmonic function of completely regular growth with respect to the proximate order $\rho(r)$, $\rho(r) \to 1$ as $r \to +\infty$, in the plane $\mathbb{C}$. Also let $W_1(r) = rW'(r)$.*

*Then the convergence of integral (23) follows from the convergence of (24) and (25) .*

P r o o f. The assertion of the theorem follows immediately from (28) and Th. 2.

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