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An Inverse Spectral Problem W.R.T. Domain

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Various practical problems, especially on hydrodynamics, elasticity theory, geophysics and aerodynamics, can be reduced to finding an optimal shape of a domain and studying its functionals.

In the paper, the inverse problem with respect to (w.r.t.) domain for two-dimensional Schrodinger operator and operator $L = \Delta^2$ is considered. The definition of s-functions is introduced. The method of determination of the domain by a given set of functions is proposed.

The main idea of the paper is to use a one-to-one correspondence between the convex bounded domains and their support functions and express the variation of the domain by the variation of corresponding support function.

Key words: Shape optimization, inverse problems, domain variation, convex domains, support function.

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1. Introduction

One of the well-studied classes of inverse problems is a class of inverse spectral problems. The papers on these problems traditionally are focused on constructing a function (potential) by given spectral data (scattering data, normalizing numbers, eigenvalues) and obtaining the necessary and sufficient conditions providing unequivocal determination of the desired function. A more detailed review can be found in [1].

There exists a wide class of practical problems that require the domain to be determined by some experimental data. For example, it is very important to find the domain of the plate under vibrations from the quantities which may be measured from distance [2]. There is a number of formulations of the inverse problem w.r.t. domain for various cases [3–5]. Note that unlike traditional inverse problems, the inverse problems w.r.t. domains have some special features. First, these problems require finding not a function, but a domain. Second, the choosing

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of data (results of measurements) that are sufficient for determination of the domain is quite a difficult problem.

In the paper we study an inverse problem w.r.t. domain for two-dimensional Schrodinger operator. In the end of the paper we put and solve a similar problem for the operator describing vibrations of the plate.

2. Problem Setting and Preliminary Results

We consider the problem

$$-\Delta u + q(x) u = \lambda u, \quad x \in D, \tag{1}$$

$$u(x) = 0, \quad x \in S_D, \tag{2}$$

where $q(x)$ is a differentiable nonnegative function satisfying the condition $t^2 q(xt) = q(x)$, $t \in R$, $0 \notin D \subset R^2$ is a bounded convex domain, $S_D \in C^2$ is its boundary, Δ is a Laplace operator.

It is known [6, p. 333] that under these conditions the eigenfunctions $u_j(x)$ of problem (1)–(2) belong to the class $C^2(D) \cap C^1(\bar{D})$, and eigenvalues λ_j are positive and, taking into account their multiplicity, may be numbered as $\lambda_1 \leq \lambda_2 \leq \ldots$.

We denote the set of all convex bounded domains $D \in R^2$ by M. Let

$$K = \left\{ D \in M : S_D \in \dot{C}^2 \right\},$$

where $\dot{C}^2$ is a class of piecewise twice continuous differentiable functions.

Definition 1. *The functions*

$$J_j(x, D) = \frac{|\nabla u_j(x)|^2}{\lambda_j}, \quad x \in D, \ j = 1, 2, \ldots \tag{3}$$

are said to be s-functions of problem (1)–(2) in the domain D, where $u_j(x)$ are normalized eigenfunctions.

We should find a domain $D \in K$ such that

$$J_j(x, D) = s_j(x), \quad x \in S_D, \ j = 1, 2, \ldots, \tag{4}$$

where $u_j(x)$ is a normalized eigenfunction and, consequently, λ_j is an eigenvalue of problem (1)–(2) in the domain D, $s_j(x)$ are given continuous functions defined on R^2.

First of all, we give some considerations that led us to this formulation.

It is important to study the dependence of eigenvalues of the operators w.r.t. domain, because mechanical characteristics of some systems indeed are eigenvalues of corresponding operators, which can be expressed by the functionals depending on domain [7, 8]. One of the steps for studying the properties of these characteristics is to compute the variation of these functionals w.r.t. domain. But to do this we should define the space of domains, give a scalar product and a definition for the domain variation in this space.

We use the following known from [12] facts:

a) For any continuous convex and positively homogeneous function $P(x)$ there exists the only convex bounded set D such that $P(x)$ is a support function of D, i.e., $P(x) = P_D(x)$. The opposite statement is also true.

b) D is found as a subdifferential of its support function at point $x = 0$

$$D = \partial P(0) = \left\{ l \in R^2 : P(x) \geq (l, x), \forall x \in R^2 \right\}.$$

It was shown [9] that the pairs $(A, B) \in M \times M$ form a linear space with the operations

$$(A, B) + (C, D) = (A + C, B + D),$$

$$\lambda(A, B) = (\lambda A, \lambda B), \quad \lambda \geq 0,$$

$$\lambda(A, B) = (|\lambda| B, |\lambda| A), \quad \lambda < 0.$$

The equivalency in $M \times M$ is defined by the relation

$$(A, B) \approx (C, D), \quad \text{if} \quad A + D = B + C.$$

The class $(0, 0)$, i.e., the set of elements (A, A), $A \in M$, plays the role of the zero element in this space. If $x = (A, B)$, then $-x = (B, A)$.

Here $A + B$ is taken in the sense of Minkowsky, i.e.,

$$A + B = \{a + b : \ a \in A, \ b \in B\}.$$

A scalar product is introduced by the following formula:

$$(a, b) = \int\limits_{S_B} P_1(x) P_2(x) ds,$$

where

$$a = (A_1, A_2), \ b = (B_1, B_2),$$

$$P_1(x) = P_{A_1}(x) - P_{A_2}(x),$$

$$P_2(x) = P_{B_1}(x) - P_{B_2}(x),$$

S_B is a unit sphere, $P_D(x) = \max\limits_{l \in D} (x, l), \ x \in R^2$ is a support function of the domain D.

We call the obtained space ML_2.

For any fixed $D \in M$, the eigenvalue λ_j of problem (1)–(2) is defined in the same way as in ([10, p. 182])

$$\lambda_j = inf I\left(u, D\right), \ \left(u, u_p\right) = 0, \ p = \overline{1, j-1},$$

where

$$I\left(u, D\right) = \frac{\int\limits_D \left[\left|\nabla u\left(x\right)\right|^2 + q\left(x\right) u^2\left(x\right)\right] dx}{\int\limits_D u^2\left(x\right) dx}.$$

Thus, we can consider λ_j as a functional of $D \in K$ and note it as $\lambda_j(D)$. The following formula is obtained (see [9, p. 98]) for the first variation of the functional $\lambda_j\left(D\right)$ in the space ML_2

$$\delta\lambda_j\left(D\right) = -\max_{u_j} \int\limits_{S_D} \left|\nabla u_j(x)\right|^2 \delta P_D\left(n(x)\right) ds, \tag{5}$$

where $\left|\nabla u_j\left(x\right)\right|^2 = \sum\limits_{i=1}^{2} \left(\frac{\partial u(x)}{\partial x_i}\right)^2$, $n(x)$ is an outward normal to S_D in the point x, max is taken over all eigenfunctions corresponding to the eigenvalue λ_j when it is multiple. Here and later on all eigenfunctions are taken normalized.

Using (5), the following formula can be obtained for the eigenvalues of (1)–(2) in the domain D

$$\lambda_j\left(D\right) = \frac{1}{2} \max_{u_j} \int\limits_{S_D} \left|\nabla u_j(x)\right|^2 P_D\left(n(x)\right) ds. \tag{6}$$

Indeed, let us take $D_0 \in K, \ D\left(t\right) = t \cdot D_0, \quad t > 0.$

By u_j we define the j-th eigenfunction of (1)–(2) corresponding to the domain D_0. Then

$$-\Delta u_j\left(x\right) + q\left(x\right) u_j\left(x\right) = \lambda_j u_j\left(x\right), \ x \in D_0.$$

This relation may be written in the following form:

$$-\frac{1}{t^2} \Delta_{\frac{x}{t}} u_j\left(\frac{x}{t}\right) + \frac{1}{t^2} q\left(\frac{x}{t}\right) u_j\left(\frac{x}{t}\right) = \frac{\lambda_j\left(D_0\right)}{t^2} u_j\left(\frac{x}{t}\right), \quad x \in D\left(t\right). \tag{7}$$

Since the function

$$\tilde{u}_j\left(x\right) = u_j\left(\frac{x}{t}\right), \ x \in D\left(t\right)$$

satisfies the relation

$$\Delta\tilde{u}_j(x) = \frac{1}{t^2} \Delta u_j\left(\frac{x}{t}\right), \tag{8}$$

then the conditions $t^2 q(tx) = q(x)$ and (8) imply

$$-\Delta \tilde{u}_j(x) + q(x)\tilde{u}_j(x) = \frac{\lambda_j(D_0)}{t^2}\tilde{u}_j(x), \ x \in D(t).$$

It shows that $\tilde{u}_j(x)$ is an eigenfunction corresponding to the eigenvalue $\lambda_j(t) = \frac{\lambda_j(D_0)}{t^2}$ of problem (1)–(2) in the domain $D(t)$. Using (5), we can write

$$\begin{gathered}\lambda_j(t+\Delta t) - \lambda_j(t) = \lambda_j(D(t+\Delta t)) - \lambda_j(D(t)) \\ = \int\limits_{S_{D(t)}} |\nabla u(x)|^2 \left[P_{D(t+\Delta t)}(n(x)) - P_{D(t)}(n(x))\right] ds + o(\Delta t), \ \ x \in S_{D(t)} \ . \end{gathered} \tag{9}$$

If the support function $P_{D(t)}(x)$ of the domain $D(t)$ is differentiable with respect to t, then dividing both sides of (9) by Δt we obtain

$$\lambda_j'(t) = -\max_{u_j} \int\limits_{S_{D(t)}} |\nabla u_j(x)|^2 P'_{D(t)}(n(x))ds, \tag{10}$$

where $P'_{D(t)}(x) = \frac{\partial}{\partial t} P_{D(t)}(x)$.

Thus

$$-2\frac{\lambda_j(D_0)}{t^3} = -\frac{1}{t^2} \underset{u_j(x)}{max} \int\limits_{S_D} \left|\nabla u_j\left(\frac{x}{t}\right)\right|^2 P_{D_0}(n(x))\, ds, \ \ x \in S_D.$$

Taking $t = 1$, we get (6).

As we can see from (6), the boundary values of function $|\nabla u_j(x)|^2$ uniquely define the eigenvalue λ_j.

From (6), taking into account (4), we obtain

$$\int\limits_{S_D} s_j(x) P_D(n(x))\, ds = 2, \quad j = 1, 2, \dots \quad . \tag{11}$$

This is the basic relation for solving the problem under consideration.

N o t e. Since we take s-functions as given data, then consider them in some concrete cases. For one-dimensional case

$$-u'' + q(x)u = \lambda u, \tag{12}$$

$$u(a) = u(b) = 0, \tag{13}$$

where $q(x) = \frac{c}{x^2}, \ c \geq 0, \ 0 \notin (a, b) \subset R$, s-functions are

$$\frac{u_{jx}^2(a)}{\lambda_j} = J_j(a),$$

$$\frac{u_{jx}^2(b)}{\lambda_j} = J_j\left(b\right).$$

Thus (11) takes the form

$$J_j\left(b\right)\cdot b - J_j\left(a\right)\cdot a = 2, j = 1, 2, \dots . \tag{14}$$

Then put $a = 0$, i.e., consider problem (12)–(13) in the interval $(0, b)$. For this case, from (9) we get the following:

Corollary. *If $a = 0$, then all s-functions of (12)–(13) satisfy the condition*

$$J_j\left(b\right) = \frac{2}{b},\ j = 1, 2, \dots . \tag{15}$$

This formula allows one to solve the inverse problem: Let a set of functions $s_j\left(x\right),\ j = 1, 2, \dots,$ be given. In this case the problem of finding a domain satisfying (4) is reduced to determining point b, what is possible by using (15).

As it was noted in the corollary, all s-functions satisfy (15) which is equivalent to the one condition. This condition is sufficient for finding point b. Indeed, we see from (15):

$$b = \frac{2}{J_j\left(b\right)}.$$

Similarly, if $b = 0$, then we have

$$a = -\frac{2}{J_j(a)}.$$

Note that if $J_j\left(x\right) \equiv c_j,\ x \in S_D,\ c_j = const,\ j = 1, 2, \dots,$ then, as it follows from (15), they all are equal to each other for all $j = 1, 2, \dots$.

In two-dimensional case, from (11) it follows that if functions $J_j(x, D)$ are constant, then

$J_j\left(x, D\right) \equiv \frac{1}{mesD},\quad j = 1, 2, \dots$ (see [9]).

Now we prove the lemma that will be used later on.

Lemma. *Let $f\left(x\right)$ be a continuous function defined on the unit sphere S_B. Then for any $D_1,\ D_2 \in K$*

$$\int\limits_{S_{D_1+D_2}} f(n(x))ds = \int\limits_{S_{D_1}} f(n(x))ds + \int\limits_{S_{D_2}} f(n(x))ds, \tag{16}$$

where $D_1 + D_2$ is taken in the sense of Minkowsky, i.e.,

$$D_1 + D_2 = \left\{x : x = x_1 + x_2,\ x_1 \in D_1,\ x_2 \in D_2\right\}.$$

P r o o f. It is known [11] that $f(x)$ may be continuously positive-homogeneously extended over the whole space and presented as a limit of difference of two convex functions

$$f(x) = \lim_{n\to\infty} \left[g_n(x) - h_n(x)\right]. \tag{17}$$

First, consider

$$f(x) = g(x) - h(x), \tag{18}$$

where $g(x)$, $h(x)$ are convex positively homogeneous functions.

As was mentioned above, there exist domains G and H such that

$$g(x) = P_G(x), \quad h(x) = P_H(x). \tag{19}$$

Considering (18), (19), we get

$$\begin{aligned} \int\limits_{S_{D_1+D_2}} f(n(x))\, ds &= \int\limits_{S_{D_1+D_2}} \left[g(n(x)) - h(n(x))\, ds\right] \\ &= \int\limits_{S_{D_1+D_2}} P_G(n(x))\, ds - \int\limits_{S_{D_1+D_2}} P_H(n(x))\, ds. \end{aligned} \tag{20}$$

For any $D_1, D_2 \in K$ the following relation is valid [9]:

$$\int\limits_{S_{D_1}} P_{D_2}(n(x))\, ds = \int\limits_{S_{D_2}} P_{D_1}(n(x))\, ds. \tag{21}$$

From (20) we obtain

$$\int\limits_{S_{D_1+D_2}} f(n(x))\, ds = \int\limits_{S_G} P_{D_1+D_2}(n(x))\, ds - \int\limits_{S_H} P_{D_1+D_2}(n(x))\, ds.$$

Since $P_{D_1+D_2}(x) = P_{D_1}(x) + P_{D_2}(x)$ [12], then, applying (21) again, we get (16).

The lemma is proved.

3. Main Results

Now we consider the main problem of the paper, that is the construction of D by a given set of functions $s_j(x)$, $j = 1, 2, \ldots$.

Let $B \subset R^2$ be a unit ball with the center at the origin and the boundary S_B. By $\varphi_k(x)$, $k = 1, 2, \ldots$, we denote some basis in $C(S_B)$-space of the functions that are continuous in S_B. These functions may be continuously positive-homogeneously extended to B

$$\tilde{\varphi}_k(x) = \begin{cases} \varphi_k\left(\frac{x}{\|x\|}\right) \cdot \|x\|, x \in B, & x \neq 0, \\ 0, \quad x = 0. \end{cases}$$

One can check that these functions are continuous and satisfy the positive homogeneity condition

$$\tilde{\varphi}_k(\alpha x) = \alpha \tilde{\varphi}_k(x), \quad \alpha > 0.$$

Without loss of generality, we can denote $\tilde{\varphi}_k(x)$ by $\varphi_k(x)$.

Thus we obtain the set of continuous positively homogeneous functions defined on B.

As we noted above, each continuous positively homogeneous function $\varphi_j(x)$ may be presented as

$$\varphi_k(x) = \lim_{n\to\infty}\left[g_n^k(x) - h_n^k(x)\right], \tag{22}$$

and there exist the domains G_n^k and H_n^k, satisfying the mentioned above properties, such that

$$g_n^k(x) = P_{G_n^k}(x), \quad h_n^k(x) = P_{H_n^k}(x).$$

We say that G_n^k and H_n^k are basic domains. Thus

$$\varphi_k(x) = \lim_{n\to\infty}\left[P_{G_n^k}(x) - P_{H_n^k}(x)\right]. \tag{23}$$

For the sake of simplicity, let us assume

$$\varphi_k(x) = P_{G^k}(x) - P_{H^k}(x), \tag{24}$$

where G^k and H^k are closed bounded convex domains.

Since $n(x) \in S_B$ for any $x \in S_D$, we can decompose $P_D(x)$ as

$$P_D(x) = \sum_{k=1}^{\infty} \alpha_k \varphi_k(x), \quad x \in S_B, \quad \alpha_k \in R, \quad k = 1, 2, \dots . \tag{25}$$

Thus, to determine $P_D(x)$ we have to find the coefficients α_k, $k = 1, 2, \dots$.

Theorem 1. *Let a set of functions $s_j(x)$, $j = 1, 2, \dots$, be given. Then the coefficients α_k, $k = 1, 2, \dots$, of the support function of the domain D, for which (4) is valid, satisfy the equation*

$$\sum_{k,m=1}^{\infty} A_{k,m}(j)\, \alpha_k \alpha_m = 2, \quad j = 1, 2, \dots , \tag{26}$$

with the coefficients

$$\begin{aligned} A_{k,m}(j) = &\int\limits_{S_{G^k}} s_j(x)\left[P_{G^m}(n(x)) - P_{H^m}(n(x))\right] ds \\ &- \int\limits_{S_{H^k}} s_j(x)\left[P_{G^m}(n(x)) - P_{H^m}(n(x))\right] ds. \end{aligned} \tag{27}$$

P r o o f. Formulas (24)–(25) imply

$$P_D(x) = \sum_{k=1}^{\infty} \alpha_k (P_{G^k}(x) - P_{H^k}(x)), x \in S_B. \tag{28}$$

Denote the set of all indexes, for which $\alpha_k \geq 0$ (resp. $\alpha_k < 0$), by I^+ (resp. I^-).
Then the relation (28) may be written as

$$\begin{aligned} &P_D\left(x\right) - \sum_{k \in I^-} \alpha_k P_{G^k}\left(x\right) + \sum_{k \in I^+} \alpha_k P_{H^k}\left(x\right) \\ &= \sum_{k \in I^+} \alpha_k P_{G^k}\left(x\right) - \sum_{k \in I^-} \alpha_k P_{H^k}\left(x\right), \quad x \in S_B. \end{aligned} \tag{29}$$

From last, taking into account the properties of support functions [12], we obtain

$$D - \sum_{k \in I^-} \alpha_k G^k + \sum_{k \in I^+} \alpha_k H^k = \sum_{k \in I^+} \alpha_k G^k - \sum_{k \in I^-} \alpha_k H^k.$$

By formula (29) and Lem. 1 we have

$$\begin{aligned} &\int\limits_{S_D} s_j(x) P_D(n(x)dx + \int\limits_{\sum\limits_{k \in I^-} (-\alpha_k) S_{G^k}} s_j(x) P_D(n(x))dx \\ &+ \int\limits_{\sum\limits_{k \in I^+} \alpha_k S_{H^k}} s_j(x) P_D(n(x)dx = \int\limits_{\sum\limits_{k \in I^+} \alpha_k S_{G^k}} s_j(x) P_D(n(x))dx \\ &+ \int\limits_{\sum\limits_{k \in I^-} (-\alpha_k) S_{H^k}} s_j(x)) P_D(n(x))dx. \end{aligned}$$

This representation and formula (11) imply

$$\begin{aligned} &\int\limits_{S_D} s_j(x) P_D(n(x)dx \\ &= \sum_{k=1}^{\infty} \alpha_k \left[\int\limits_{S_{G^k}} s_j\left(x\right) P_D(n(\xi))dx - \int\limits_{S_{H^k}} s_j(x) P_D(n(x)dx \right] = 2. \end{aligned}$$

Substituting (25) into this formula we obtain (26) with the coefficients (27). The theorem is proved.

We assumed that the problem considered had a solution in general case. For some interesting cases, where the functions $s_j(x)$, $j = 1, 2, \ldots$, are defined as experimental data, this problem always has a solution. The function $P_D(x)$ is constructed with the help of (26) using (25).

As noted above, the domain D is uniquely defined by its support function $P_D(x)$. Suppose that (26) has the only solution providing convexity of the support function of D.

Let us show that the expressions $\frac{|\nabla u_j(x)|^2}{\lambda_j}$, $j = 1, 2, \ldots$, for (1)–(2) in the constructed with the help of formula (25) domain D are s-functions. Indeed, if $\bar{D}$ is a domain, in which problem (1)–(2) has s-functions given by formula (25), then representing $\overline{D}$ by formulae (25) we get the equation (26) with the same coefficients. From the assumption that this equation has the only solution it follows that $\overline{D} = D$.

If (16) has more than one solution, then the desired domain is one of those constructed by (18) using these solutions, providing convexity of $P(x)$.

This algorithm is constructed under the assumption (24). In a general case, when $\varphi(x)$ has the form of (22), $A_{k,m}(j)$ turns into

$$A_{k,m}(j) = \lim_{n\to\infty} \left[\int\limits_{S_{G_n^k}} s_j(x) \left[P_{G_n^m}(n(x)) - P_{H_n^m}(n(x)) \right] ds \right.$$

$$\left. - \int\limits_{S_{H_n^k}} s_j(x) \left[P_{G_n^m}(n(x)) - P_{H_n^m}(n(x)) \right] ds \right].$$

Now consider the transverse vibrations of the plate.

Let $D \in R^2$ be a domain of the plate with the boundary $S_D \in C^2$.

It is known [2] that the function $\omega\ (x_1 x_2,\, t)$ describing the transverse vibrations of the plate satisfies the equation

$$\omega_{x_1x_1x_1x_1} + 2\omega_{x_1x_1x_2x_2} + \omega_{x_2x_2x_2x_2} + \omega_{tt} = 0. \tag{30}$$

Assuming the process stabilized, we look for a solution — eigenvibration in the form

$$\omega(x_1, x_2, t) = u(x_1, x_2)\, cos\lambda t,$$

where λ is an eigenfrequency.

Substituting this representation into (30), we arrive to

$$\Delta^2 u = \lambda u, \quad x \in D, \tag{31}$$

where $\Delta^2 = \Delta\Delta$.

For various cases various boundary conditions may be given. In the case under consideration we deal with a squeezed plate with the boundary conditions

$$u = 0, \quad \frac{\partial u}{\partial n} = 0, \quad x \in S_D. \tag{32}$$

Let

$$K = \left\{ D \in M : S_D \in \dot{C}^2 \right\},$$

where $\dot{C}^2$ is a class of piecewise twice continuous differentiable functions.

Definition 2. *The functions* $J_j\left(x, D\right) = \frac{|\Delta u_j(x)|^2}{\lambda_j}$, $x \in R^2$, $j = 1, 2, \ldots$, *are called s-functions of (31)–(32) in the domain* D.

We should find a domain $D \in K$ such that

$$J_j\left(x, D\right) = s_j(x), x \in S_D, j = 1, 2, \ldots, \tag{33}$$

where $u_j\left(x\right)$ is an eigenvibration, and λ_j is an eigenfrequency of (31)–(32) in the domain D, $s_j\left(x\right)$, $j = 1, 2, \ldots$, are given continuous functions defined in R^2.

In [9], for the eigenfrequency of the squeezed plate under transverse vibrations the following formula is obtained :

$$\lambda_j = \frac{1}{4} \max_{u_j} \int\limits_{S_D} \left|\Delta u_j(x)\right|^2 P_D(n(x)) ds, \tag{34}$$

where $P_D\left(x\right) = \underset{l \in D}{max}\left(l, x\right), \quad x \in R^2$ is a support function of D, and max is taken over all eigenvibrations u_j corresponding to eigenfrequency λ_j. (As we see from (33), the boundary values of the function $\left|\Delta u_j\left(x\right)\right|^2$ uniquely define λ_j). From (34) and (33) we get

$$\int\limits_{S_D} s_j\left(x\right) P_D\left(n\left(x\right)\right) ds = 4, \;\; j = 1, 2, \ldots .$$

From the above consideration we can conclude that the following theorem is proved.

Theorem 2. *Let a set of functions $s_j(x)$, $j = 1, 2, \dots,$ be given. Then the coefficients of support function of domain D of the plate, for which (33) is valid, satisfy the equation*

$$\sum_{k,m=1}^{\infty} A_{k,m}(j)\,\alpha_k \alpha_m = 4, \quad j = 1, 2, \dots ,$$

with the coefficients

$$A_{k,m}(j) = \lim_{n\to\infty} \left[\int\limits_{S_{G_n^k}} s_j(x) \left[P_{G_n^m}(n(x)) - P_{H_n^m}(n(x)) \right] ds \right.$$

$$\left. - \int\limits_{S_{H_n^k}} s_j(x) \left[P_{G_n^m}(n(x)) - P_{H_n^m}(n(x)) \right] ds \right].$$

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