Journal of Mathematical Physics, Analysis, Geometry
2008, vol. 4, No. 1, pp. 108–120

On the Simon–Spencer Theorem

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Received October 20, 2007

This paper presents a generalization of the classical result by B. Simon and T. Spencer on the absence of absolutely continuous spectrum for the continuous one-dimensional Schrödinger operator with an unbounded potential.

Key words: Schrödinger operator, localization, Simon–Spencer theorem.
Mathematics Subject Classification 2000: 81Q10, 47E05, 34L40.

Dedicated with great respect to V. Marchenko and L. Pastur

1. Introduction

The fundamental paper by B. Simon and T. Spencer (see [9]) has played an essential role in our understanding of localization phenomena. For the lattice Schrödinger operator, the main result of this paper is quite transparent and can be formulated in the following form:

Theorem 1.1. *Let $h = \Delta + V(x)$, $x \geq 0$ be the lattice Schrödinger operator on $l^2(\mathbb{Z}_+)$ with the boundary condition $\psi(0) = 0$. If*

$$\limsup_{x\to\infty} |V(x)| = \infty,$$

then $\sum_{ac}(h) = \emptyset$.

R e m a r k. Due to general results (see [5]) concerning compact perturbations of h, $\sum_{ac}(h) = \emptyset$ for any boundary condition of the form $\psi(-1)\cos\theta - \psi(0)\sin\theta = 0$ with $\theta \in [0, \pi)$.

The result cannot be improved. There are many examples of operators h with bounded potentials $V(x)$ whose spectra are either purely absolutely continuous, or contain a rich absolutely continuous component. For instance, for periodic V the spectrum $\sum(h)$ of h is purely absolutely continuous. This statement is physically nontrivial for energies in the range of the potential V.

For the continuous Hamiltonian, the corresponding result is not so strong, and the result depends on the existence of very high "peaks" in the potential function V.

Theorem 1.2. *Let $\mathcal{H}_+\psi = -\psi'' + V\psi$ be a 1-D Schrödinger operator on $L^2(\mathbb{R}_+)$ with the Dirichlet boundary condition $\psi(0) = 0$ and $V(x) \geq 0$. If there exist sequences $\{x_n\}_{n\geq 0}$, $\{h_n\}_{n\geq 0}$ and $\{\delta_n\}_{n\geq 0}$ of positive numbers with x_n, $h_n \to \infty$ for which $V(x) \geq h_n$ on $[x_n, x_n + \delta_n]$ and $\delta_n\sqrt{h_n} \to \infty$, then $\sum_{ac}(\mathcal{H}_+) = \emptyset$.*

Theorem 1.2 does not cover the physically significant class of "δ-like" potentials. We can expect that for potential functions of the type

$$V(x) = \sum_{n\geq 1} h_n \delta(x - x_n) \quad \text{or} \quad V(x) = \sum_{n\geq 1} h_n \frac{\mathbb{I}_n(x)}{\delta_n}$$

(here $\mathbb{I}_n$ represents the indicator function of the interval $[x_n, x_n + \delta_n]$) for which $x_n - x_{n-1} \to \infty$, $h_n \to \infty$ and $\delta_n \to 0$, the corresponding Hamiltonian $\mathcal{H}_+$ will have no absolutely continuous component. However, Th. 1.2 cannot be used at all to prove this for the δ-potential shown above, and requires a strong assumption in the second case, namely $\sqrt{h_n\delta_n} \to \infty$. Our goal is to prove the following result generalizing Th. 1.2 in several directions:

Theorem 1.3. *Let $\mathcal{H}$ be a one-dimensional Schrödinger operator on $L^2(\mathbb{R})$ defined by*

$$\mathcal{H} = -\frac{d^2}{dx^2} + V(x). \tag{1}$$

Assume that $V(x) \geq 0$ and that

$$\limsup_{|x|\to\infty} \int\limits_x^{x+1} V(z)\ dz = \infty. \tag{2}$$

Then $\sum_{ac}(\mathcal{H}) = \emptyset$.

A similar statement is true for the half axis case.

Theorem 1.4. *Let $\mathcal{H}_\theta$ be a 1-D continuous Schrödinger operator on $L^2(\mathbb{R}_+)$ with the boundary condition $\psi(0)\cos\theta - \psi'(0)\sin\theta = 0$, $\theta \in [0,\pi)$. Assume that $V(x) \geq 0$ and that*

$$\limsup_{x\to\infty} \int\limits_x^{x+1} V(z)\ dz = \infty. \tag{3}$$

Then $\sum_{ac}(\mathcal{H}_\theta) = \emptyset$.

R e m a r k 1. Of course, (3) implies that $\limsup_{x\to\infty} \int_x^{x+\omega} V(z)\ dz = \infty$ for any $\omega > 0$.

R e m a r k 2. All final or "nearly final" results in spectral theory contain local L^1 norms of the potential. We remind the reader of the following results (see [2]) of M. Birman and A. Molchanov. M. Birman proved that the spectrum of $\mathcal{H}$ is bounded from below if and only if

$$\limsup_{x\to\infty} \int\limits_x^{x+1} V_-(s)\ ds < \infty, \tag{4}$$

where $V_-(x) = \max(0, -V(x))$. Moreover, if

$$\lim_{x\to\infty} \int\limits_x^{x+1} V_-(s)\ ds = 0, \tag{5}$$

then the negative spectrum is purely discrete (possibly with an accumulation point at 0). Additionally, if $V \leq 0$, then condition (5) is also necessary.

Another result was given by A. Molchanov: if $V \geq 0$, then the spectrum of $\mathcal{H}$ is purely discrete if and only if for any $\omega > 0$

$$\lim_{|x|\to\infty} \int\limits_x^{x+\omega} V(s)\ ds = \infty.$$

In the second part of the paper, we will present an example showing that the condition

$$\limsup_{x\to\infty} \int\limits_x^{x+1} V(z)\ dz = \infty,$$

together with self-adjointness of $\mathcal{H}$, cannot guarantee the absence of the absolutely continuous spectrum. In fact, in this example the absolutely continuous spectrum will coincide with $[0,\infty)$. The key feature of this example will be the presence of very deep wells which tend to destroy the repulsive effects caused by high positive peaks.

Finally, we will consider the Hamiltonian $\mathcal{H}$ with the potential

$$V(x) = \sum_{n\geq 1} h_n \delta(x - x_n) \tag{6}$$

and prove the following theorem:

Theorem 1.5. *Let $\delta > 0$ and $\mathcal{H}_\theta$ be the operator on $L^2(\mathbb{R}_+)$ defined by $\mathcal{H}_\theta\psi(x) = -\psi''(x) + V(x)\psi(x)$ with the boundary condition $\psi(0)\cos\theta - \psi'(0)\sin\theta = 0$, where $V(x)$ is defined by (6) with $h_n = n$.*

(a) If $x_n - x_{n-1} > (n!)^{2+\delta}$, then the spectrum of $\mathcal{H}_\theta$ is purely singular continuous for any boundary phase $\theta \in [0,\pi)$.

(b) If $x_n - x_{n-1} < (n!)^{2-\delta}$, then the spectrum of $\mathcal{H}_\theta$ is pure point for a.e. $\theta \in [0,\pi)$.

2. A Few Lemmas and the Proof of Theorem 1.3

Following the strategy of Simon and Spencer ([9]), we first want to study the following problem: let

$$H \equiv \mathcal{H} + I = -\frac{d^2}{dx^2} + V(x) + 1, \tag{7}$$

where $V(x) \geq 0$ for all $x \in \mathbb{R}$. Suppose that

$$\int_{-L}^{L} V(s)\ ds = A >> 1. \tag{8}$$

We want to estimate (for the energy parameter $\lambda = 0$) the trace norm $||H^{-1} - H_{x_0}^{-1}||_1$ of the difference of the resolvents of the operators H and H_{x_0}. Here, H_{x_0} is the operator given by the differential expression $-d^2/dx^2 + V(x) + 1$ with the Dirichlet boundary condition $\psi(x_0) = 0$ at some point $x_0 \in [-L, L]$.

Lemma 2.1. *The kernel $H_\lambda^{-1}(x,y)$ of the resolvent operator $(H-\lambda)^{-1}$ at the point $\lambda = 0$ has the following representation:*

$$\begin{aligned} H_0^{-1}(x,y) &\equiv R(x,y) \\ &= \mathbb{E}_x \left\{ \int_0^\infty \exp\left(-\int_0^t (1 + V(b_s))\ ds \right) \delta_y(b_t)\ dt \right\}. \end{aligned}$$

Here, b_t is the Brownian motion with the generator $\mathcal{L} = d^2/dx^2$.

This is one of the well-known forms of the Feynman–Kac formula (see [6]) connecting the Schrödinger operator (outside its spectrum) with the Brownian motion. The expression $\delta_y(b_t)\ dt$ takes the form $\delta_y(b_t)\ dt = d\Upsilon_y(t)$, where $\Upsilon_y(t)$ is the local time of b_s at the point y.

Lemma 2.2. *If $x, y < x_0$ or $x, y > x_0$, then*

$$R_{x_0}(x,y) = \mathbb{E}_x \left\{ \int_0^{\tau_0} \exp\left(-\int_0^t (1+V(b_s))\ ds \right) \delta_y(b_t)\ dt \right\}, \tag{9}$$

where τ_0 is the time of the first arrival of the Brownian motion b_t at the point x_0, that is, $\tau_0 = \min\{t : b_t = 0\}$.

R e m a r k. Of course, $\tau_0 < \infty$ with probability one.
A similar result is true in a more general situation.

Lemma 2.2′. *Let $R_{x_0,X}$ be the resolvent (again for $\lambda = 0$) of the operator H defined by the expression (7) and Dirichlet boundary conditions at x_0 and at each point of a discrete set $X \subset \mathbb{R}$ ($x_0 \notin X$). Then*

$$R_{x_0,X}(x,y) = \mathbb{E}_x \left\{ \int_0^{\tau_0 \wedge \tau_X} \exp\left(-\int_0^t (1+V(b_s))\ ds \right) \delta_y(b_t)\ dt \right\}. \tag{10}$$

Here, both x and y belong to one of the intervals Δ_i, where $\{\Delta_i : i = 1, 2, \ldots\}$ is the partition of $\mathbb{R}$ by the point x_0 and the points of X. The random moment τ_X is defined by

$$\tau_X = \min\{t : b_t \in X\}.$$

R e m a r k. Since $\tau_0 \geq \tau_0 \wedge \tau_X$, from (9) and (10) it follows that $R_{x_0,X}(x,x) \leq R_{x_0}(x,x)$ on each interval Δ_i. This monotonicity property will be used in the proof of Th. 1.3.

Lemma 2.3. *If $x, y < x_0$ or $x, y > x_0$, then*

$$\begin{aligned} R(x,y) - R_{x_0}(x,y) &= \mathbb{E}_x \left\{ \exp\left(-\int_0^{\tau_0} (1+V(b_s))\ ds \right) \right\} \\ &\times \mathbb{E}_{x_0} \left\{ \int_0^{\infty} \exp\left(-\int_0^t (1+V(b_s))\ ds \right) \delta_y(b_s)\ ds \right\} \\ &= \psi_\pm(x) R(x_0, y), \end{aligned} \tag{11}$$

where $\psi_+(x)$ is used in (11) for $x, y < x_0$ and $\psi_-(x)$ for $x, y > x_0$. Here, $\psi_+(x)$ is the solution of $H\psi = 0$ on $(-\infty, x_0]$, such that $\psi_+(x_0) = 1$ and $\psi_+(x) \to 0$ as $x \to -\infty$. Similarly, $\psi_-(x)$ is the solution of $H\psi = 0$ on $[x_0, \infty)$ satisfying $\psi_-(x_0) = 1$ and $\psi_-(x) \to 0$ as $x \to \infty$. Such solutions exist and are unique. Furthermore, these solutions are positive, monotone and convex over the intervals $(-\infty, x_0]$ and $[x_0, \infty)$, respectively.

R e m a r k. It is easy to see that $\psi_+(x) \le e^{-|x-x_0|}$ for $x \le x_0$ and $\psi_-(x) \le e^{-|x-x_0|}$ for $x \ge x_0$.

P r o o f. We have

$$\begin{aligned}
R(x,y) - R_{x_0}(x,y) &= \mathbb{E}_x \left\{ \int_0^\infty \exp\left(-\int_0^t (1 + V(b_s))\, ds \right) \delta_y(b_t)\, dt \right\} \\
&- \mathbb{E}_x \left\{ \int_0^{\tau_0} \exp\left(-\int_0^t (1 + V(b_s))\, ds \right) \delta_y(b_t)\, dt \right\} \\
&= \mathbb{E}_x \left\{ \int_{\tau_0}^\infty \exp\left(-\int_0^{\tau_0} (1 + V(b_s))\, ds \right) \right. \\
&\times \left. \exp\left(-\int_{\tau_0}^t (1 + V(b_s))\, ds \right) \delta_y(b_t)\, dt \right\}. \qquad (12)
\end{aligned}$$

Using the strong Markov property for the stopping time τ_0, we then have

$$\begin{aligned}
R(x,y) - R_{x_0,X}(x,y) &= \mathbb{E}_x \left\{ \exp\left(-\int_0^{\tau_0} (1 + V(b_s))\, ds \right) \right\} \\
&\times \mathbb{E}_{b_{\tau_0}} \left\{ \int_0^\infty \exp\left(-\int_0^u (1 + V(b_s))\, ds \right) \delta_y(b_u)\, du \right\} \\
&= R(x_0, y) \mathbb{E}_x \left\{ \exp\left(-\int_0^{\tau_0} (1 + V(b_s))\, ds \right) \right\}.
\end{aligned} \qquad (13)$$

We have used the obvious relation $b_{\tau_0} = x_0$. The elliptic form of the Feynman–Kac formula gives for

$$u(x) = \mathbb{E}_x \left\{ \exp\left(-\int_0^{\tau_0} (1 + V(b_s))\, ds \right) \right\}$$

the equation (for $x < x_0$ or $x > x_0$)

$$u'' - (1 + V)u = 0, \quad u(x_0) = 1,$$

i.e., $u(x)$ is equal to $\psi_+(x)$ for $x \le x_0$, or $\psi_-(x)$ for $x \ge x_0$. ∎

As with Lem. 2.2, Lem. 2.3 can be generalized.

Lemma 2.3′. *The difference $R(x, y) - R_{x_0,X}(x, y)$ is given by the expression*

$$\begin{aligned} & R(x, y) - R_{x_0,X}(x, y) \\ = \;& R(x_0, y)\mathbb{E}_x \left\{ \exp\left(- \int\limits_0^{\tau_0 \wedge \tau_X} (1 + V(b_s))\delta_y(b_s)\; ds \right) \right\}, \end{aligned}$$

where x, y belong to the same interval Δ_i.

R e m a r k. Lemmas 2.3 and 2.3′ contain fundamental information about $R(x, y) - R_{x_0,X}(x, y)$ and $R(x, y) - R_{x_0}(x, y)$. Both of these differences are non-negative and increase if we replace the potential V by a smaller function, say by the truncated potential $V(x)\mathbb{I}_\Delta(x)$, where $\mathbb{I}_\Delta$ is the indicator function of an arbitrary interval Δ, or remove extra Dirichlet boundary conditions imposed at points of the set X. In particular,

$$R(x, y) - R_{x_0}(x, y) \le R(x, y) - R_{x_0,X}(x, y)$$

(compare with the remark following Lem. 2.2′).

In the following paragraph and in Lem. 2.4, we denote by $V_\Delta(x)$ the truncated potential $V_\Delta(x) = V(x)\mathbb{I}_\Delta(x)$, where Δ is the interval $[-L, L]$. Let H_Δ be the operator

$$H_\Delta = -d^2/dx^2 + 1 + V_\Delta(x)$$

and $R_\Delta = H_\Delta^{-1}$.

Using the functions $\psi_\pm(x)$ given by Lem. 2.3, in which $V(x)$ is replaced by $V_\Delta(x)$, we can construct the resolvent kernel $R_\Delta(x_0, x)$, i.e., the L^2 solution of the problem $H_\Delta R_\Delta = -\delta_{x_0}$, namely

$$R_\Delta(x_0, x) = \begin{cases} c\psi_+(x) & \text{if} \quad x < x_0 \\ c\psi_-(x) & \text{if} \quad x > x_0 \end{cases}, \tag{14}$$

where the constant c is such that

$$c(\psi'_+(x_0) - \psi'_-(x_0)) = 1. \tag{15}$$

It also follows that

$$c = \frac{\tilde{\psi}'_+(x_0)}{\tilde{\psi}_+(x_0)} - \frac{\tilde{\psi}'_-(x_0)}{\tilde{\psi}_-(x_0)},$$

where $\tilde{\psi}_{\pm}(x)$ are arbitrary solutions of $H_\Delta\psi = 0$, exponentially decaying at $\mp\infty$, respectively. For example, we can define $\tilde{\psi}_+(x) = e^x$ for $x < -L$ and $\tilde{\psi}_-(x) = e^{-x}$ for $x > L$. From (14) it then follows (see [9]) that

$$\begin{aligned} \|H_\Delta^{-1} - H_{\Delta,x_0}^{-1}\|_1 &= \operatorname{Tr}\left(H_\Delta^{-1} - H_{\Delta,x_0}^{-1}\right) \\ &= \int_{-\infty}^{\infty} \left(R_\Delta(x,x) - R_{\Delta,x_0}(x,x)\right)\,dx \\ &= c\left(\int_{-\infty}^{x_0} \psi_+(x)^2\,dx + \int_{x_0}^{\infty} \psi_-(x)^2\,dx\right). \end{aligned}$$

Let us note that $0 \le \psi_+(x) \le 1$ and that $\psi_+(x) \le e^x$ for $x < -L$, so that

$$\int_{-\infty}^{x_0} \psi_+(x)^2\,dx \le \int_{-\infty}^{-L} e^{2x}\,dx + \int_{-L}^{x_0} 1\,dx \le \frac{1}{2} + 2L. \tag{16}$$

Similarly,

$$\int_{x_0}^{\infty} \psi_-(x)^2\,dx \le \frac{1}{2} + 2L. \tag{17}$$

Now, we are ready to prove the central technical result.

Lemma 2.4. *For an appropriate $x_0 \in [-L, L]$*

$$\|H_\Delta^{-1} - H_{\Delta,x_0}^{-1}\|_1 = \operatorname{Tr}(R_\Delta - R_{\Delta,x_0}) \le \frac{c(L)}{\sqrt{A}}, \tag{18}$$

where $A = \int_{-L}^{L} V(s)ds \ge 1$ and $c(L)$ is some constant depending only on L.

P r o o f. Let us introduce the phase function

$$z(x) = \tilde{\psi}'(x)/\tilde{\psi}(x),$$

where $\tilde{\psi}(x)$ is the solution of $H\psi = 0$ satisfying the boundary conditions $\psi(-L) = \psi'(-L) = 1$. Then $z(x) = 1$ for $x \in (-\infty, -L]$, since $\tilde{\psi}(x) = e^x$ on this interval. The function $z(x)$ satisfies the usual Riccati equation

$$z'(x) = (1 + V(x)\mathbb{I}_\Delta(x)) - z(x)^2, \quad z(-L) = 1, \tag{19}$$

where $\mathbb{I}_\Delta$ is the indicator of the interval $\Delta = [-L, L]$. After integration, (19) becomes

$$z(x) = 1 + \int_{-L}^{x} (1 + V(s)\mathbb{I}_\Delta(s))\,ds - \int_{-L}^{x} z(s)^2\,ds. \tag{20}$$

Put $M = \max_{x\in\Delta} z(x)$. Since $z(x)$ is continuous, there is a minimal point $x_0 \in [-L, L]$ for which $M = z(x_0)$. Then, (20) implies $M \geq z(L) \geq 1+2L+A-2LM^2$, which in turn gives $M + 2LM^2 \geq A$. It follows that

$$M \geq b(L)\sqrt{A}$$

if $A \geq 1$, where $b(L) > 0$ depends only on L. Now, since $\psi'_-(x) < 0$ and $\psi_-(x) > 0$, by (15) it follows that $1 = c(\psi'_+(x_0) - \psi'_-(x_0)) \geq c\psi'_+(x_0)$, and therefore putting $c(L) = 1/b(L)$,

$$c \leq \frac{1}{\psi'_+(x_0)} = \frac{\tilde{\psi}_+(x_0)}{\tilde{\psi}'_+(x_0)} = \frac{1}{z(x_0)} = \frac{1}{M} \leq \frac{c(L)}{\sqrt{A}}.$$

■

Now we are ready to prove Th. 1.3.

P r o o f o f T h e o r e m 1.3. For fixed $L > 0$, let $\Delta_n = [y_n - L, y_n + L]$, $n \in \mathbb{Z}$, be a sequence of disjoint intervals for which $y_n \to \pm\infty$ as $n \to \pm\infty$, and

$$\sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{A_n}} < \infty, \tag{21}$$

where $A_n = \int_{\Delta_n} V(s)\, ds$. Using Lem. 2.4, one can find a point $x_{0,n} \in \Delta_n$ for which

$$||H^{-1}_{\Delta_n} - H^{-1}_{\Delta_n,x_{0,n}}||_1 \leq \frac{c(L)}{\sqrt{A_n}}. \tag{22}$$

Here, $H_{\Delta_n} = -d^2/dx^2 + 1 + V(x)\mathbb{I}_{\Delta_n}(x)$ and $H_{\Delta_n,x_{0,n}}$ is the same operator, but with Dirichlet boundary condition added at $x_{0,n} \in \Delta_n$. Now, let us return to the operator $\mathcal{H}$ defined by (1), and consider the resolvents $(\mathcal{H} + 1)^{-1}$ and $(\mathcal{H}_X + 1)^{-1}$, where $\mathcal{H}_X$ is the operator $\mathcal{H}$ with Dirichlet boundary conditions at the countable system of points $X = \{x_{0,n}\}$. Using the fact that both $\mathcal{H}$ and $\mathcal{H}_{x_0,X}$ are nonnegative, it follows from the monotonicity argument (see the remark following Lem. 2.3′) that

$$||(\mathcal{H} + 1)^{-1} - (\mathcal{H}_X + 1)^{-1}||_1 \leq \sum_n \frac{c(L)}{\sqrt{A_n}} < \infty.$$

By the Kato–Birman theorem (applicable since $\lambda = -1$ is outside the spectrum of both operators, $\mathcal{H}$ and $\mathcal{H}_X$, see ([8])), it follows that $\sum_{ac}(\mathcal{H}) = \sum_{ac}(\mathcal{H}_{x_0,X})$. But the operator $\mathcal{H}_X$ is the orthogonal sum of the operators $\mathcal{H}_n$, where $\mathcal{H}_n = -d^2/dx^2 + V(x)$ on the interval $[x_{0,n}, x_{0,n+1}]$ with Dirichlet boundary conditions at the endpoints. Since each $\mathcal{H}_n$ has purely discrete spectrum, the spectrum of $\mathcal{H}_X$ is pure point. Therefore $\sum_{ac}(\mathcal{H}) = \emptyset$. ■

3. A Few Examples

The first example will show that the presence of a strong positive part of the potential cannot guarantee the absence of the absolutely continuous spectrum of $\mathcal{H}$, even when $\mathcal{H}$ is essentially selfadjoint.

Example 3.1. *Let $x_n, h_n \to \infty$, $\delta_n \to 0$, $x_{n+1} - x_n \to \infty$ and $\mathcal{H} = -d^2/dx^2 + V(x)$, where*

$$V(x) = \sum_{n=1}^{\infty} h_n(\mathbb{I}_{[x_n-\delta_n,x_n]} - \mathbb{I}_{[x_n,x_n+\delta_n]}). \tag{23}$$

Note that, from the conditions of Ex. 3.1,

$$\int_{x_n-1}^{x_n} V(x)\, ds = \delta_n h_n \to \infty$$

and that $\mathcal{H}$ is essentially selfadjoint. This follows by results due to P. Hartman and M. Eastham (see [4, 1]) giving the essential self-adjointness of $\mathcal{H}$, without any assumption on V other than that $V(x) \geq 0$ on some infinite disjoint sequence of intervals of fixed length.

Theorem 3.2. *If in Ex. 3.1 $\sum_n h_n^2 \delta_n^3 < \infty$, then $\sum_{ac}(\mathcal{H}) = [0, \infty)$.*

It will be helpful to consider the monodromy matrix M_λ in the generalized Prüfer representation, that is, $M_\lambda(a,b)$ is the matrix satisfying

$$\begin{bmatrix} \psi(b) \\ \frac{\psi'(b)}{\sqrt{\lambda}} \end{bmatrix} = M_\lambda(a,b) \begin{bmatrix} \psi(a) \\ \frac{\psi'(a)}{\sqrt{\lambda}} \end{bmatrix}.$$

Lemma 3.3. *Let $V_{\delta,h} = h(\mathbb{I}_{[-\delta,0]} - \mathbb{I}_{[0,\delta]})$ and $M_\lambda(-\delta,\delta)$ be the monodromy matrix in the generalized Prüfer representation for the problem $\mathcal{H} = -d^2/dx^2 + V_{\delta,h}(x)$ on $[-\delta,\delta]$. Let Δ be a fixed interval on the positive energy axis and suppose $\lambda \in \Delta$. Then, with the assumption $\delta << 1$ and $h >> 1$,*

$$||M_\lambda(-\delta,\delta) - I|| \leq ch^2\delta^3.$$

P r o o f. Assume that $h >> \lambda > 0$ and let $\alpha_{h,\delta}(\lambda) = \sqrt{h-\lambda}\,\delta$. Let us write an explicit formula for $M_\lambda(-\delta,0)$ and $M_\lambda(0,\delta)$. Simple calculations show that

$$\begin{aligned} M_\lambda(-\delta,0) &= \begin{pmatrix} \cosh \alpha_{h,\delta}(\lambda) & \frac{\sqrt{\lambda}\delta}{\alpha_{h,\delta}(\lambda)} \sinh \alpha_{h,\delta}(\lambda) \\ \frac{\alpha_{h,\delta}(\lambda)}{\sqrt{\lambda}\delta} \sinh \alpha_{h,\delta}(\lambda) & \cosh \alpha_{h,\delta}(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} 1 + O(h\delta^2) & O(\delta) \\ \frac{h\delta}{\sqrt{\lambda}} + O(h^2\delta^3) & 1 + O(h\delta^2) \end{pmatrix} \end{aligned} \tag{24}$$

while

$$M_\lambda(0,\delta) = \begin{pmatrix} \cos\alpha_{h,\delta}(\lambda) & \frac{\sqrt{\lambda}\delta}{\alpha_{h,\delta}(\lambda)}\sin\alpha_{h,\delta}(\lambda) \\ -\frac{\alpha_{h,\delta}(\lambda)}{\sqrt{\lambda}\delta}\sin\alpha_{h,\delta}(\lambda) & \cos\alpha_{h,\delta}(\lambda) \end{pmatrix} = \begin{pmatrix} 1+O(h\delta^2) & O(\delta) \\ -\frac{h\delta}{\sqrt{\lambda}}+O(h^2\delta^3) & 1+O(h\delta^2) \end{pmatrix}. \tag{25}$$

With the fact that $M_\lambda(-\delta,\delta) = M_\lambda(0,\delta)M_\lambda(-\delta,0)$, it follows from (24) and (25) that

$$||M_\lambda(-\delta,\delta) - I|| = O(h^2\delta^3), \tag{26}$$

which implies $||M_{\lambda,n} - I|| \le Ch_n^2\delta_n^3$, where $M_{\lambda,n} = M_\lambda(x_n-\delta_n, x_n+\delta_n)$. Now $M_\lambda(0,x_n+\delta_n) = O_nM_{\lambda,n}\cdots O_2M_{\lambda,2}, O_1M_{\lambda,1}$, where O_i are appropriate orthogonal matrices, and from (26) it follows that

$$||M_\lambda(0,x_n+\delta_n)|| \le \prod_{k=1}^n(1+Ch_k^2\delta_k^3) \le \exp\left(\sum_{k=1}^n Ch_k^2\delta_k^3\right) < \infty.$$

It is known that the existence of a sequence x_n, for which the monodromy matrix is uniformly bounded from above for all energies in a fixed interval Δ, implies the absolute continuity of the spectrum in this interval (see [7]). We have proved that $\sum_{ac}(\mathcal{H}) \supseteq [0,\infty)$. In fact, it is easy to prove that $\sum_{ac}(\mathcal{H}) = [0,\infty)$. ■

The second example is related to the one above. We will use here and in Ex. 3.4 the following observation: let $\mathcal{H} = -d^2/dx^2 + h\delta_0(x)$. Then, in the generalized Prüfer representation

$$M_\lambda(0-,0+) = \begin{pmatrix} 1 & 0 \\ \frac{h}{\sqrt{\lambda}} & 1 \end{pmatrix}. \tag{27}$$

Example 3.4. *Let $V(x)$ be the potential defined by*

$$V(x) = \sum_n h_n(\delta(x-x_n) - \delta(x-x_n-\delta_n)),$$

where $h_n, x_n \to \infty$ and $\delta_n \to 0$. Let $\mathcal{H}_\theta$ be defined on $L^2(\mathbb{R}_+)$ by $\mathcal{H}_\theta = -d^2/dx^2 + V(x)$ with the boundary condition $\psi(0)\cos\theta - \psi'(0)\sin\theta = 0$ with $\theta \in [0,\pi)$.

From (27), an explicit formula for $M_\lambda(x_n-0, x_n+\delta_n+0)$ can be obtained, namely

$$M_\lambda(x_n-0,x_n+\delta_n+0) = \begin{pmatrix} 1 & 0 \\ \frac{h_n}{\sqrt{\lambda}} & 1 \end{pmatrix} O_\lambda(x_n-\delta_n-0,x_n+0) \begin{pmatrix} 1 & 0 \\ -\frac{h_n}{\sqrt{\lambda}} & 1 \end{pmatrix}, \tag{28}$$

where

$$O_\lambda(x_n-\delta_n-0,x_n+0)=\\ \begin{pmatrix} \cos\sqrt{h_n-\lambda}\delta_n & \frac{1}{\sqrt{h_n-\lambda}}\sin\sqrt{h_n-\lambda}\delta_n \\ -\sqrt{h_n-\lambda}\sin\sqrt{h_n-\lambda}\delta_n & \cos\sqrt{h_n-\lambda}\delta_n \end{pmatrix}. \tag{29}$$

From (28) and (29), one can deduce that

$$||M_\lambda||=1+O(h_n\delta_n^2),$$

and hence if

$$\sum_n \delta_n^2 h_n<\infty,$$

then $\sum_{ac}(H_\theta)=[0,\infty)$ for any $\theta\in[0,\pi)$.

Example 3.5. *Let $V(x)=\sum h_n\delta(x-x_n)$, where $h_n,x_n>0$ and $h_n,x_n\to\infty$. Let $\mathcal{H}_\theta$ be defined on $L^2(\mathbb{R}_+)$ by $\mathcal{H}_\theta=-d^2/dx^2+V(x)$ with the boundary condition $\psi(0)\cos\theta-\psi'(0)\sin\theta=0$ with $\theta\in[0,\pi)$.*

Since $h_n\to\infty$, it follows immediately from Th. 1.4 that $\sum_{ac}(\mathcal{H})=\emptyset$. We can estimate the norm by

$$||M_\lambda(0,x_n+0)||=\left\|\prod_{k=1}^n\begin{pmatrix}1 & 0\\ \frac{h_k}{\sqrt{\lambda}} & 1\end{pmatrix}\right\|\le c(\lambda)^n\prod_{i=1}^n h_i$$

and in general $||M_\lambda(0,x+0)||=\left\|\prod_{k=1}^n\begin{pmatrix}1 & 0\\ \frac{h_k}{\sqrt{\lambda}} & 1\end{pmatrix}\right\|\le c(\lambda)^{n(x)}\prod_{i=1}^{n(x)}h_i$ with $n(x)=\#\{x_i|x_i\le x\}$. Now,

$$\int_0^\infty \frac{dx}{||M_\lambda(0,x)||^2}\ge\sum_i\frac{x_i-x_{i-1}}{h_1^2h_2^2\cdots h_{n(x_i)}^2c(\lambda)^{2n(x_i)}}.$$

For fast increasing distances x_i-x_{i-1} and fixed h_i the last series diverges, from which it follows (see [10]) $\sum_{pp}(\mathcal{H})=\emptyset$. In this particular case the spectrum is purely singular continuous. It is probably the simplest example of an operator with purely singular continuous spectrum (compare [3] and [10]). In fact, if $h_n=n$ and $x_n=(n!)^{2+\delta}$ for $\delta>0$, then $\sum(\mathcal{H}_\theta)=\sum_{sc}(\mathcal{H})$. One can prove also that for $x_n=(n!)^{2-\delta}$ with $\delta>0$, the spectrum of $\mathcal{H}_\theta$ is pure point for a.e $\theta\in[0,\pi)$. This proves Th. 1.5 formulated in the introduction.

References

[1] *M. Eastham*, On a Limit-Point Method of Hartman. — *Bull. London Math. Soc.* **4** (1972), 340–344.

[2] *I. Glazman*, Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators. Israel Progr. for Sci. Transl., Jerusalem, 1965.

[3] *A. Gordon, S. Molchanov, and B. Tsagani*, Spectral Theory for One-Dimensional Schrödinger Operators with Strongly Fluctuating Potentials. — *Funct. Anal. Appl.* **25** (1992), 236–238.

[4] *P. Hartman*, The Number of L^2-Solutions of $x'' + q(t)x = 0$. — *Amer. J. Math.* **43** (1951), 635–645.

[5] *T. Kato*, Perturbation Theory for Linear Operators (2nd Ed.). Springer–Verlag, Berlin, Heidelberg, 1995.

[6] *H. McKean*, Stochastic Integrals. Acad. Press, New York, 1969.

[7] *S. Molchanov*, Multiscale Averaging for Ordinary Differential Equations. Homogenization Series on Advances in Mathematics for Applied Sciences. — *World Sci.* **50** (1999), 316–397.

[8] *M. Reed and B. Simon*, Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness. Acad. Press, London, 1975.

[9] *B. Simon and T. Spencer*, Trace Class Perturbations and the Absence of Absolutely Continuous Spectrum. — *Comm. Math. Phys.* **87** (1982), 253–258.

[10] *B. Simon and G. Stolz*, Operators with Singular Continuous Spectrum, V. Sparse Potentials. — *Amer. Math. Soc.* **124** (1996), 2073–2080.