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On Principle of Equicontinuity

(Recommended by Prof. E. Dshalalow)

The main purpose of this paper is to prove some results of uniform boundedness principle type without the use of Baire's category theorem in certain topological vector spaces; this provides an alternate route and important technique to establish certain basic results of functional analysis. As applications, among other results, versions of the Banach—Steinhaus theorem and the Nikodym boundedness theorem are obtained.

Обоснованы некоторые результаты типа принципа однородной ограниченности без использования теоремы категории Байера в некоторых топологических векторных пространствах, что обеспечивает альтернативный способ и важную методику для получения результатов функционального анализа. Получены также версии теоремы Банаха — Штейнхауза и теоремы ограниченности Никодима.

Key words: principle of equicontinuity, Banach—Steinhaus theorem, locally convex space, thick set.

1. Introduction. The classical uniform boundedness principle asserts: if a sequence $\{f_n\}$ of continuous linear transformations from a Banach space X into a normed space Y is pointwise bounded, then $\{f_n\}$ is uniformly bounded. The proof of this result is most often based on the Baire's category theorem (e. g. see Theorem 4.7—3 [18] and Theorem 3.17 [26]); the interested reader is referred to [10] for a new approach in this context. Several authors have sought proof of this type of results without Baire's theorem in various settings (see, for example, [4, 16], [23], [27]).

In 1933, Nikodym [21] proved: If a family M of bounded scalar measures on a σ-algebra **A** is setwise bounded, then the family M is uniformly bounded. This result is a striking improvement of the uniform boundedness principle in the space of countably additive measures on **A**; a Baire category proof of this theorem may be found in [9, IV. 9.8, p. 309]. Nikodym theorem has received a great deal of attention and has been generalized in several directions [5, 8, 19, 20, 28];

in particular, the proofs of this result without category argument for finitely additive measures with values in a Banach space (quasi-normed group) are provided by Diestel and Uhl [6, Theorem 1, p. 14] and Drewnoski [7, Theorem 1], respectively. For other related generalizations of this theorem, we refer to the bibliography in [6].

Recently, Nygaard [22, 23] has used the notion of a «thick set» to prove the uniform bounded principle for transformations on a thick subset of a Banach space X with values in another Banach space Y. The concept of a thick set goes back to the ideas of Kadets and Fonf [12, 13, 15]. It is worth pointing out that the concept of «thick sets» heavily depends on the dual of X and the development of their theory essentially relies on the Hahn—Banach separation theorem in X. The broader class of «thick sets» contains as a subclass the class of second category sets.

In this paper, certain aspects of the development of the uniform boundedness principle are discussed; in particular, results of the type of uniform boundedness principle are proved on a domain of second category and beyond without employing Baire's category argument. First, we prove a general principle of equicontinuity for maps on a topological vector space of the second category with values in another topological vector space. A similar result is obtained for transformations on «thick sets» of a complete locally convex space X satisfying the property (N) and taking values in a locally convex space Y; this generalizes the uniform boundedness principle of Nygaard [23] to a class of locally convex spaces. An analogue of the new result is given for maps from X^* into Y^*. Some versions of the Banach—Steinhaus theorem and the Nikodym boundedness theorem are also given.

2. Notations and preliminaries. Let P be a family of seminorms on a Hausdorff locally convex space X. Let $B_X = \{x \in X : p(x) \le 1 \text{ for each } p \in P\}$ and $S_X = \{x \in X : p(x) = 1 \text{ for each } p \in P\}$; see [3, part III, p. 13, 14)]. The strong dual X^* of X is a locally convex space (details may be found in [3, part IV, p. 14—23]. For our purposes, it would be enough to consider the following: suppose that Ω is a family of bounded subsets of X. The pair $(\Omega, |\cdot|)$ induces a locally convex topology on X^* via the family P^* of seminorms

$$p^*(x^*) = \sup\{|x^*(x)| : x \in A, A \in \Omega\}.$$

Similarly, if Q is a family of seminorms on a locally convex space Y, then Q^* will be the induced family of seminorms defining the locally convex topology on Y^*.

Let X and X^* be in duality. The polar of $A \subset X$ and $B \subset X^*$ are, respectively, defined by

$$A^0 = \left\{x^* \in X^* : \sup_{x \in A} |x^*(x)| \le 1\right\},$$

$$B^0 = \{x \in X\colon \sup_{x^* \in B} |x^*(x)| \le 1\},$$

where we consider X to be embedded in X^{**}, bidual of X [30].

Locally convex spaces provide a very general framework for the Hahn—Banach theorem and its consequences; in particular, we shall need the following separation result.

Proposition 2.1 [27, Prop. 13, p. 173]. Let A be a closed and absolutely convex subset of a Hausdorff locally convex space X and $x \notin A$. Then there exists $x^* \in X^*$ such that $|x^*(x)| > 1 \ge \sup\{x^*(y) : y \in A\}$.

In what follows we will use the terminology of Nygaard [22, 23].

A subset A of a normed space X is norming for X^* if for some $\delta > 0$, $\inf_{x^* \in S_{X^*}} \sup_{x \in A} |x^*(x)| \ge \delta$. Analogously, a subset B of X^* is norming for X (or ω^*-norming) if for some $\delta > 0$, $\inf_{x \in S_X} \sup_{x^* \in B} |x^*(x)| \ge \delta$. We say a subset A of X is thin if it is countable union of an increasing sequence of sets which are non-norming for X^*. A set which is not thin, is called a thick set.

The concept of ω^*-thin and ω^*-thick sets can be defined in the same way.

A set A in a complex vector space X is norming if for some $\delta > 0$, $\overline{co}\left(\bigcup_{|r|=1} rA\right) \supset \delta B_X$. However, we shall employ $\overline{co}(\pm A) \supseteq \delta B_X$ for simplicity.

It will be interesting to formulate the above definitions in the context of an arbitrary locally convex space.

Let G be a commutative group. A non-negative valued function q on G is said to be a quasi-norm if it has the following properties for any x, y in G:

(i) $q(0) = 0$;

(ii) $q(x) = q(-x)$;

(iii) $q(x + y) \le q(x) + q(y)$.

The relationship of functional analysis and measure theory is not so easy to understand (for some connections, we refer to [14]). Recently, Abrahamsen et al. [1] have established in Prop. 3.2, boundedness of a vector measure by utilizing the concept of a thick set; thereby reflecting growing interaction between these two subjects. Consequently, such an interplay will play a part here.

Let G be a commutative Hausdorff topological group and $\mathbf{R}$ a ring of subsets of a set X. A function $\mu : \mathbf{R} \to G$ is said to be: (i) measure if $\mu(\phi) = 0$ and $\mu(E \cup F) = \mu(E) + \mu(F)$ where E and F are in $\mathbf{R}$ with $E \cap F = \phi$ (ii) exhaustive if for every sequence $\{E_n\}$ of pairwise disjoint sets in $\mathbf{R}$, $\lim_{n\to\infty} \mu(E_n) = 0$.

The notion of a submeasure has been extensively studied by Drewnoski [7, 8 and the references therein]. The applications of this concept are enormous [8, 24]. Group-valued submeasures have been introduced by Khan and Rowlands [17] and their work has been further investigated by Avallone and Valente [2].

Let G be a commutative lattice group (ℓ-group). A quasi-norm q on G is an ℓ-quasi-norm if $q(x) \le q(y)$ for all x, y in G with $|x| \le |y|$ where $|x| = x^+ + x^-$. An ℓ-quasi-norm generates a locally solid group topology on G [14, Prop. 2 2C]. Following Khan and Rowlands [17], a G-valued function μ on $\mathbf{R}$ is a submeasure if $\mu(\phi) = 0$, $\mu(E \cup F) \le \mu(E) + \mu(F)$ for all E, F in $\mathbf{R}$ with $E \cap F = \phi$ and $\mu(E) \le$ $\le \mu(F)$ for all E, F in $\mathbf{R}$ with $E \subseteq F$. Clearly, in this case $\mu(E) \ge 0$ for all E in $\mathbf{R}$.

3. Main results. Khan and Rowlands [16] have obtained the following improvement of Theorem 2 due to Daneš [4].

Theorem A. [16, Corollary 1]. Let X be a topological vector space, $\{x_n\}$ a sequence in X such that $\lim\limits_{n\to\infty} x_n = 0$, and $\{p_n\}$ a sequence of real sub-additive functionals on X satisfying the condition: «there exists a sequence $\{a_k\}$ of real numbers, $a_k \to +\infty$ as $k \to \infty$, such that, for each $k, n = 1, 2, \ldots$, the set $B_{k,n} = \{x \in$ $\in X : p_n(x) \le a_k\}$ is closed in X».

If $\lim\limits_n \sup(\sup\limits_{x\in U} p_n(x)) = +\infty$ for each neighbourhood U of 0 in X, then the set $Z = \{z \in X : \lim\limits_n \sup p_n(x_n + z) = +\infty$ or $\lim\limits_n \sup p_n(x_n - z) = +\infty\}$ is a residual G_δ-set in X.

The following example reveals that Theorem A is not true, in general, if Z is replaced by either Z^+ or Z^- where $Z^+ = \{x \in X : x > 0\}$, $Z^- = \{x \in X : x < 0\}$ and $Z = Z^+ \cup Z^-$.

Let X be the usual space of real numbers. We assume that $x_n = 0$ for each $n \in \mathrm{N}$. Define $p_n(x) = n|x|$ $(x \in X,\ n \in \mathrm{N})$. Here $Z^- = \phi$ and so Z^- can not be residual G_δ-set while Z^+ is a residual G_δ-set, for $X \setminus Z^+ = \{0\}$ is of first category in X. Thus, either Z^+ or Z^- can be a residual G_δ-set.

As an application of Theorem A, we establish a principle of equicontinuity in the following result; this leads to an alternative proof of the Banach—Steinhaus theorem given by Rudin [25].

Theorem 1. (Principle of equicontinuity). Let X be a topological vector space of the second category, Y a Hausdorff topological vector space and $\{f_n\}$ a sequence of continuous linear transformations of X into Y such that the set $\{f_n(x)\}$ is bounded for each $x \in X$. Then the sequence $\{f_n\}$ is equicontinuous.

P r o o f. Let the topology of Y be determined by a family $\{q_i : i \in I\}$ of F-seminorms (definition and details may be found in [29, p. 1—3]). Suppose that the sequence $\{f_n\}$ is not equicontinuous. Then for some continuous quasi-norm

q_{i_0}, which for the sake of simplicity we denote by q, and any τ-neighbourhood U of 0 in X, there exist a sequence $\{x_n\}$ in U and a sequence of integers $n_{k_1} < n_{k_2} < n_{k_3} < \cdots$ such that $q(f_{n_k}(x_n)) > k$ ($k = 1, 2, ...$). It follows that $\lim_n \sup(\sup_{x \in U} q(f_n(x))) = +\infty$. The functionals $q_0 f_n$ ($n = 1, 2, ...$) satisfy the conditions of Theorem A (taking $x_n = 0$ for all $n = 1, 2, ...$), and so the set

$$Z = \{z \in X : \lim_n \sup q(f_n(z)) = +\infty\}$$

is a residual G_δ-set in X. Thus $X \setminus Z$ is of the first category. Since X is of the second category, it follows that Z is non-empty; this implies that there is a point $z_0 \in X$ such that $\lim_n \sup q(f_n(z_0)) = +\infty$. This contradicts the hypothesis. Thus the sequence $\{f_n\}$ is equicontinuous.

An immediate consequence of the above theorem is given below.

Theorem 2. (Banach—Steinhaus theorem). Let X and Y be as in Theorem 1 and let $\{f_n\}$ be a sequence of continuous linear transformations of X into Y such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in X$. Then f is a continuous linear transformation of X into Y.

P r o o f. Clearly, f is a homomorphism and the sequence $\{f_n(x)\}$ is bounded. By Theorem 1, the sequence $\{f_n\}$ is equicontinuous. Let V be any neighbourhood of 0 in Y. Then there exist a closed neighbourhood $V_0 \subseteq V$ and a neighbourhood U of 0 in X such that $f_n(U) \subseteq V_0$ ($n = 1, 2, ...$). Now, for any $x \in U$,

$$f(x) = \lim_{n\to\infty} f_n(x) \in \overline{V}_0 = V_0$$

and so $f(U) \subseteq V$; that is, f is continuous.

For X, as in Theorem 1, let $M \subset X^*$ be ω^*-bounded (i.e., $\sup\{|f(x)| : f \in M\} < \infty$ for every $x \in X$). Then M is pointwise bounded in X^* and so bounded by Theorem 1.

In the same way, some other results purely dependent on the classical uniform boundedness principle can be adopted from [11, 26, 30] in this general setting.

As another application of Theorem A, we indicate how the Banach—Steinhaus theorem on condensation of singularities [27, Corollary 3, p. 121] may be derived from it.

Theorem 3. Let $\{U_{n,m} : n, m = 1, 2, ...\}$ be a double sequence of bounded linear transformations of a Banach space X into a Banach space Y such that for each $m = 1, 2, ...$, $\lim_n \sup \|U_{n,m}\| = +\infty$. Then there is a set S of the second category in X such that, for each x in S and each $m = 1, 2, ...$, $\lim_n \sup \| U_{n,m}(x) \| = +\infty$.

P r o o f. For each positive integer m, n define $p_{n,m}(x) = \| U_{n,m}(x) \| (x \in X)$. It is easy to see that each $p_{n,m}$ is a continuous sub-additive functional on X. For each positive integer m, define

$$Z_m = \{z \in X : \limsup_n p_{n,m}(z) = +\infty\}$$

and

$$Z = \bigcap_{m=1}^{\infty} Z_m.$$

The condition $\limsup_n \|U_{n,m}\| = +\infty$ implies that, for each m = 1, 2, ..., $\limsup_n (\sup_{x \in U} p_{n,m}(x)) = +\infty$ for each neighbourhood U of 0 in X, and therefore by Theorem A (with $x_n = 0$ for all positive integers n), Z_m is a residual G_δ-set. It follows that Z is a residual G_δ-set. Since X is a Banach space, therefore $Z = \{z \in X : \limsup_n \| U_{n,m}(z) \| = +\infty$ for $m = 1, 2, ...\}$ is of second category and is the desired set S.

A locally convex space in which a norm is available, is said to have the property (N). For example, a normed space and the space (X^*, ω^*) where X is a locally convex space have the property (N).

In the remainder of this section it is assumed that X is a complete locally convex space with the property (N).

We need the following pair of lemmas.

L e m m a 1. The following statements are equivalent for a subset A of X:

a) A is norming for X^*;

b) $\overline{co}\,(\pm A)$ is norming for X^*;

c) there exits a $\delta > 0$ such that $\overline{co}\,(\pm A) \supset \delta B_X$.

P r o o f. The only non-trivial implication is (a) $\Rightarrow$ (c).

Assume that $\overline{co}\,(\pm A) \subset \delta B_X$ for all $\delta > 0$. Consider a sequence $\{x_n\}$ in $X \setminus \overline{co}\,(A)$ converging to 0. For each n, $x_n \notin \overline{co}\,(\pm A)$, an absolutely convex subset of X, so by Prop. 2.1 (see also Theorem 4.25 in [11]) there exists $x_n^* \in X^*$ such that

$$|x_n^*(x_n)| > \sup_{a \in \overline{co}(\pm A)} |x_n^*(a)| \geq \sup_{a \in A} |x_n^*(a)| .$$

Now using (a), we may obtain $\delta > 0$ satisfying

$$|x_n^*(x)| > \inf_{x_n^* \in S_{X^*}} \sup_{a \in A} |x_n^*(a)| > \delta .$$

Plainly the choice of $\{x_n\}$ implies that $|x_n^*(x_n)| < \delta$ for all $\delta > 0$ and $n \geq n_0$. This contradiction proves the result.

The following analogous result for the dual space X^* is easy to verify.

L e m m a 2. The following statements are equivalent for a subset B of X^*:

a) B is norming for X;

b) $\overline{co}\,(\pm B)$ is norming for X;

c) there exists $\delta > 0$ such that $\overline{co}^{\,\omega^*}(\pm B) \supseteq \delta B_{X^*}$.

L e m m a 3. If A is a subset of the second category in X, then A is thick.

P r o o f. Let $\{A_i\}$ be an increasing sequence with $A = \bigcup_{i=1}^{\infty} A_i$. As A is of second category, some $\overline{A_m}$ contains a ball $S_r(x)$. Hence, it follows that $S_r(0) \subseteq \overline{co}\,(\pm A_m)$. This implies, by Lemma 1 (with $\delta = 1$), A_m is norming. Since $\{A_i\}$ is arbitrary, therefore A must be thick.

The classical uniform boundedness principle holds beyond sets of the second category; this is the case with the set S of characteristic functions in the unit sphere of the function space $B(\mathbf{A})$ where $\mathbf{A}$ is a σ-algebra of sets [6]. Note that S is merely nowhere dense. We continue this theme and generalize Theorems 1 and 2 and Prop. 2.2 of Nygaard [23] in the sense that the domain of transformations is a thick set in X and its dual space X^*. Our methods are based on those used by Nygaard [22, 23].

Theorem 4. Let A be a thick subset of X. Suppose that Y is a Hausdorff locally convex space and $\{f_n\}$ a sequence of continuous linear transformations of X into Y such that $\{f_n(x)\}$ is bounded for each $x \in A$. Then the sequence $\{f_n\}$ is equicontinuous.

P r o o f. Suppose that $\{f_n\}$ is pointwise bounded on A, that is, $\sup_n p(f_n(x)) < \infty$ for all $x \in A$ and each $p \in P$. Put $A_m = \{x \in A : \sup_n p(f_n(x)) \le m$ for each $p \in P\}$. The sequence $\{A_m\}$ of sets is increasing with $A = \bigcup_{i=1}^{\infty} A_i$. As A is thick, some A_k is norming. Thus, by Lemma 1, there exists a $\delta > 0$ such that $\delta B_X \subseteq \overline{co}\,(\pm A_k)$. This together with the definition of A_m implies that $\delta p(f_n) =$

$$= \sup_{x \in \delta S_X} p(f_n(x)) \le \sup_{x \in \overline{co}(\pm A_k)} p(f_n(x)) \le k.$$ Hence, $\sup_n p(f_n) \le \frac{k}{\delta} < \infty$ as desired.

R e m a r k. Theorem 4 extends Prop. 2.2 of Nygaard [23].

Theorem 5. Let B be a thick subset of X^*. Suppose that Y is a Hausdorff locally convex space and $\{f_n^*\}$ a sequence of continuous linear transformations of X^* into Y^* such that $\{f_n^*(x^*)\}$ is bounded for each x^* in B. Then the sequence $\{f_n^*\}$ is equicontinuous.

P r o o f. Follows pattern of the proof of Theorem 4; the only difference is that we consider

$$A_m = \{x^* \in B : \sup_n q^* (f_n^* (x^*)) \le m \text{ for each } q^* \in Q^*\}$$

and use Lemma 2 and the ω^*-continuity of f_n^*.

The proofs of the following corollaries follow pattern of the proof of Theorem 2 and so will be omitted.

Corollary 1. Let X, A and Y be as in Theorem 4 and $\{f_n\}$ be a sequence of continuous linear transformations of X into Y such that $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in X$. Then f is a continuous linear transformation of X into Y.

Corollary 2. Let X^*, B and Y^* be as in Theorem 5 and $\{f_n^*\}$ be a sequence of continuous linear transformations of X^* into Y^* such that $f^*(x^*) = \lim_{n\to\infty} f_n^*(x^*)$ exists for each $x^* \in X^*$. Then f^* is a continuous linear transformation of X^* into Y^*.

We now establish the Nikodym boundedness theorem in more general settings in relation to the domain, range and nature of mappings.

Theorem 1 due to Drewnoski [7] is proved in the context of a quasi-normed group; we observe that his proof can be readily modified to the case of any commutative Hausdorff topological group G to obtain a principle of equicontinuity type result for group measures as follows.

Theorem 6. Let M be a family of exhaustive G-valued measures on a σ-ring $\mathbf{R}$ such that for each $E \in \mathbf{R}$, $\{\mu(E) : \mu \in M\}$ is a bounded subset of G. Then $\{\mu(E) : E \in \mathbf{R}, \mu \in M\}$ is a bounded subset of G.

The assumption that $\mathbf{R}$ is a σ-ring is essential in the above theorem [7, Example, p. 117].

Valuable contributions have been made in special but very important field of submeasures with values in a commutative ℓ-group [2, 17 and the references therein]. In the next result, we prove Theorem 6 for group-valued submeasures to get the following principle of equicontinuity which generalizes Theorem 1 of Drewnowski [7].

Theorem 7. Let (G, q) be an ℓ-quasi-normed group and M be a family of G-valued submeasures on a σ-ring $\mathbf{R}$ such that

$$\sup_{\mu \in M} q(\mu(E)) < +\infty$$

for each E in $\mathbf{R}$. Then $\sup_{\substack{\mu \in M \\ E \in \mathbf{R}}} q(\mu(E)) < +\infty$.

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P r o o f. Let H be the group of all G-valued mappings on M. Clearly, H is a commutative partially ordered group, the order being $f \le g$ if and only if $f(\mu) \le$ $\le g(\mu)$ for all $\mu \in M$. Define the functional ϕ on H by

$$\phi(f) = \sup_{\mu \in M} q(f(\mu)) .$$

Note that ϕ is an extended real-valued quasi-norm on H with $\phi(f) \le \phi(g)$ for $0 \le f \le g$. Define a mapping $\nu : \mathbf{R} \to H$ by

$$\nu(E)(\mu) = \mu(E) .$$

Clearly, ν is an H-valued submeasure on $\mathbf{R}$.

Suppose not; then with the above notation, $\sup_{E \in \mathbf{R}} \phi(\nu(E)) = +\infty$. Thus, for each positive integer n, there exists a set E_n in $\mathbf{R}$ such that $\phi(\nu(E_n)) > n$. Let $E = \bigcup_{n=1}^{\infty} E_n$. Now $E \in \mathbf{R}$ and $\phi(\nu(E)) = +\infty$. This implies that $\sup_{\mu} q(\mu(E)) = +\infty$, which contradicts the hypothesis. Hence, $\sup_{\substack{\mu \in M \\ E \in \mathbf{R}}} q(\mu(E))$ is finite.

Finally, every σ-algebra of sets on a finite set S is a topology but not conversely. Thus, the result to follow extends the domain of maps in [6, Corollary 2, p. 16] and [23, Prop. 2.1], simultaneously.

Theorem 8. Let A be a thick subset of X. If $\{f_n\}$ is a sequence of continuous linear functionals on X such that $\{f_n(x)\}$ is bounded for each x in A, then the sequence $\{f_n\}$ is equicontinuous.

P r o o f. Take Y as the space of scalars in the proof of Theorem 4.

Обгрунтовано деякі результати типу принципу однорідної обмеженості без використання теореми категорії Байєра у деяких топологічних векторних просторах, що забезпечує альтернативний спосіб та важливу методику для отримання результатів функціонального аналізу. Отримано також версії теореми Банаха—Штейнхауза та теореми обмеженості Нікодима.

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