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МЕТОДЫ И МОДЕЛИ

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Filtering of an Inventory Model with a Multinomial Thinning Operator

(Recommended by Prof. E. Dshalalow)

In this paper a multivariate discrete-time, discrete-state stochastic inventory model for perishable items is discussed. This model draws on earlier works by the authors and the fractional thinning operator of Steutel and van Harn. Items in stock are assumed to belong to one of M possible categories (representing qualities). At each time t items in the stock may stay in the same class, move to one of the $M-1$ classes or perish. The movement between classes is assumed to be regulated by a multinomial thining operator (to be defined below) which is dependent on some vector-valued parameter process. Recursive estimates for the parameter process are proposed for three possible scenarios.

Рассмотрена стохастическая модель управления запасами с многими случайными переменными, дискретная во времени и пространстве, для скоропортящихся товаров. Модель построена на основе предыдущих работ авторов с использованием дробного оператора разрежения Стентела и Ван Харна. Предполагается, что товары на складе относятся к одной из M возможных категорий качества. В каждый момент времени t товары на складе могут оставаться в одном и том же классе, переходить в один из $M-1$ классов или портиться. Предполагается также, что перемещение между классами регулируется мультиномиальным оператором разрежения, который зависит от некоторого процесса с векторно-оцениваемыми параметрами. Для трех возможных сценариев предложены рекурсивные оценки параметров процесса.

K e y w o r d s: partially observed inventory model, multinomial thinning operator, optimal filtering.

1. Introduction. Deterioration (perishability) of items while in stock is a real fact. Food, electronic components, pharmaceuticals, and drugs are just a few examples of such items: see [1—3]. In this paper we consider a multivariate discrete state, discrete time stochastic inventory model for perishable items, where items are assumed to belong to $M+1$ possible categories (representing qualities). Categories are assumed to be ordered so that Category 1 houses the best quality and quality M houses the pre-perished quality and the perished items are housed in Category $M+1$. At each time t, $t = 1, \ldots$ items in Category i, $i = 1,..., M$, that have not been sold either stay in the same class, or move to a lower class. The movement between classes is regulated by some multinomial thinning operator to be defined below. As a matter of fact the proposed model builds on an earlier work by the authors where the Binomial thinning operator « $\circ$ » is used: see [4, 5] .

To Binomial thinning operator is defined as follows. For any nonnegative integer-valued random variable X and $\alpha \in [0,1]$, let

$$\alpha \circ X = \sum_{j=1}^{X} Y_j,$$

where Y_1, Y_2, ... is a sequence of of i. i. d. random variables independent of X, such that $P(Y_j = 1) = 1 - P(Y_j = 0) = \alpha$. We assume that $\alpha \circ X = 0$ if $X \leq 0$. The operator « $\circ$ » was used by [6, 7] to examine integer-valued time series and to model count data.

Here we shall assume that the inventory consists of a single item. Let X_n^i, $i = 1, ..., M+1$, $n = 1, ...$ be the level of stock of the item in category i, at time n. We also assume that within period n, an item of quality i either keeps its quality with probability α_i^i or move to any of the $M-i$ lower qualities $i+1$, $i+2$, ..., M with probabilities $\alpha_{i+1}^i, \alpha_{i+2}^i, ..., \alpha_M^i$, or perish with probability $1-\alpha_{i+1}^i - \alpha_{i+2}^i - ... - \alpha_M^i \triangleq \alpha_{M+1}^i$.

The inventory dynamics now take the form

$$\begin{aligned} X_n^1 &= \alpha_1^1 \circ X_{n-1}^1 + U_n - V_n^1, \\ X_n^2 &= \alpha_2^1 \circ X_{n-1}^1 + \alpha_2^2 \circ X_{n-1}^2 - V_n^2, \\ &\vdots \\ X_n^M &= \alpha_M^1 \circ X_{n-1}^1 + \alpha_M^2 \circ X_{n-1}^2 + ... + \alpha_M^M \circ X_{n-1}^M - V_n^M, \\ X_n^{M+1} &= \alpha_{M+1}^1 \circ X_{n-1}^1 + \alpha_{M+1}^2 \circ X_{n-1}^2 + ... + \alpha_{M+1}^M \circ X_{n-1}^M, \end{aligned} \tag{1}$$

where U is a $\boldsymbol{Z}_+$-valued process representing the replenishment process which is assumed to be predictable with respect to the filtration generated by the inventory process, and

$$\boldsymbol{V}_n \triangleq (V_n^1, V_n^2, ..., V_n^M, 0)' \tag{2}$$

is a $\boldsymbol{Z}_+^M$ -valued random variable with distribution ϕ_n representing the demand at each epoch n.

Now we introduce the multinomial operator ◊. This operator can be seen as a natural generalization of the binomial thinning. Let X be an non-negative, integer-valued random variable. For a given value of X, suppose that a random experiment consists of classifying each of X objects into one of $M+1$ categories with probabilities $\alpha_1, \alpha_2, \ldots, \alpha_M, \alpha_{M+1}$ such that $\alpha_1 + \alpha_2 + \ldots + \alpha_M + \alpha_{M+1} = 1$. Write

$$\boldsymbol{S}_X = \left\{ \boldsymbol{\ell} = (\ell_1, \ldots, \ell_{M+1})' : \sum_{i=1}^{M+1} \ell_i = X \right\},$$

where $\ell_1, \ldots, \ell_{M+1}$ are nonnegative integers. Let

$$\boldsymbol{\alpha} \stackrel{\Delta}{=} (\alpha_1, \alpha_2, \ldots, \alpha_{M+1})'. \tag{3}$$

Then $\boldsymbol{\alpha} \lozenge X = \sum_{\ell \in S_X} \boldsymbol{\ell} I(\boldsymbol{Y} = \boldsymbol{\ell})$, where $\boldsymbol{Y} = (Y^1, \ldots, Y^{M+1})'$, Y^i is the (random) number of objects that result in class i.

Then with e_i denoting the M-dimensional standard unit vector with 1 in the i-th position and zeroes elsewhere, the dynamics in (1) take the form

$$\boldsymbol{X}_n = \sum_{i=1}^{M} \boldsymbol{\alpha}^i \lozenge \langle \boldsymbol{X}_{n-1}, e_i \rangle + \boldsymbol{U}_n - \boldsymbol{V}_n, \tag{4}$$

where

$$\boldsymbol{X}_n = \begin{bmatrix} X_n^1 \\ X_n^2 \\ \vdots \\ X_n^{M+1} \end{bmatrix}, \quad \boldsymbol{U}_n = \begin{bmatrix} U_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \boldsymbol{\alpha}^1 = \begin{bmatrix} \alpha_1^1 \\ \alpha_2^1 \\ \vdots \\ \alpha_M^1 \\ \alpha_{M+1}^1 \end{bmatrix}, \quad \boldsymbol{\alpha}^2 = \begin{bmatrix} 0 \\ \alpha_2^2 \\ \vdots \\ \alpha_M^2 \\ \alpha_{M+1}^2 \end{bmatrix}, \ldots, \quad \boldsymbol{\alpha}^M = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \alpha_M^M \\ \alpha_{M+1}^M \end{bmatrix},$$

$\boldsymbol{V}_n$ is given in (2).

A generalization of the operator ◊ [8] was proposed in [9]. In both articles this operator or its generalization gave rise to some new integer-valued time series where their properties were examined. The present paper take a different approach and focusses on estimating dynamically the parameters $\alpha^1, \ldots, \alpha^M$. To do that we shall consider three possible scenarios. Initially, we shall adopt a Bayesian point of view and assume that parameters $\alpha^1, \ldots, \alpha^M$ have some given prior density. Based on observing the level of stock of the items, recursive estimate for the posterior density is proposed. This is done in the next section. In the second scenario, it is assumed that the parameters $\alpha^1, \ldots, \alpha^M$ are no longer static in time but dynamic and changes according to some Markovian rule. Also, based on observing the inventory history recursive estimates are proposed in section 3. The final scenario builds on the fact that the assumption of full observation of the level of stock is not always valid as transaction errors, spoilage, product quality and yield render full observation of the level of stock difficult. In this paper, we shall consider the zero balance walk proposed by [10, 11], where at each period demand is only observed when the inventory level drops to zero. The paper concludes with the analysis of the zero-balance walk model and some general remarks.

2. Recursive parameter estimation. In this section we derive recursive estimates for the parameters $\boldsymbol{\alpha}^1, ..., \boldsymbol{\alpha}^M$. We suppose that each $\boldsymbol{\alpha}^i$ takes values in a measurable space $(\Theta^i, \beta^i, \mu^i)$. The values of $\boldsymbol{\alpha}^1, ..., \boldsymbol{\alpha}^M$ are unknown and, in this section, we suppose they are constant.

Write $\mathcal{F}_n$ for the complete history generated by the observed inventory $\boldsymbol{X}_k$, $k = 0, 1, \ldots, n$, and $\mathcal{G}_n$ for the complete history generated by the inventory and the parameters $\boldsymbol{\alpha}^1, ..., \boldsymbol{\alpha}^M$.

To make computations easy: see [12] we shall work under a reference probability measure $\overline{P}$ where the process $\boldsymbol{X}$ is a sequence of i.i.d. random variables with probability distribution ψ. Set $\lambda_0 = 1$, and for $k \geq 1$

$$\lambda_k = \frac{\phi_n\left(\sum_{i=1}^{M} \boldsymbol{\alpha}^i \lozenge \langle \boldsymbol{X}_{n-1}, e_i \rangle + \boldsymbol{U}_n - \boldsymbol{X}_n\right)}{\psi(\boldsymbol{X}_n)}, \quad \Lambda_n = \prod_{k=0}^{n} \lambda_k .$$

It can be shown that the process $\{\Lambda_n\}$ is a martingale with respect to the filtration $\mathcal{G}_n$. Therefore, we can relate P and $\overline{P}$ by setting $\frac{dP}{d\overline{P}}\Big|_{\mathcal{G}_n} \triangleq \Lambda_n$.

We shall call the probability measure P the «real world» measure. Using similar arguments to those used in [12], we can also show that under the «real world» measure P that the dynamics in (4) hold where $\boldsymbol{V}_n \triangleq \sum_{i=1}^{M} \boldsymbol{\alpha}^i \lozenge \langle \boldsymbol{X}_{n-1}, e_i \rangle +$ $+ \boldsymbol{U}_n - \boldsymbol{X}_n$, and under P, $\boldsymbol{V}_n$ has distribution ϕ_n.

Now, from observing the inventory level, we are interested in computing $E\left[\prod_{i=1}^{M} I(\boldsymbol{\alpha}^i \in d\boldsymbol{\theta}^i) | \mathcal{F}_n\right]$. A generalized version of Bayes Theorem: see [12] gives:

$$E\left[\prod_{i=1}^{M} I(\boldsymbol{\alpha}^i \in d\boldsymbol{\theta}^i) | \mathcal{F}_n\right] = \frac{\overline{E}\left[\Lambda_n \prod_{i=1}^{M} I(\boldsymbol{\alpha}^i \in d\theta^i) | \mathcal{F}_n\right]}{\overline{E}\left[\Lambda_n | \mathcal{F}_n\right]} .$$

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The numerator of the above expression represents a unnormalized conditional expectation. Write

$$\overline{E}\left[\Lambda_n \prod_{i=1}^{M} I(\boldsymbol{\alpha}^i \in d\boldsymbol{\theta}^i) | \mathcal{F}_n\right] \triangleq q_n(\boldsymbol{\theta}^1, ..., \boldsymbol{\theta}^M)\ d\mu^1(\boldsymbol{\theta}^1) \dots d\mu^M(\boldsymbol{\theta}^M).$$

The normalizing denominator $\overline{E}\,[\Lambda_n | \mathcal{F}_n]$ is given by

$$\int_{\Theta^1 \times \dots \times \Theta^M} q_n(\boldsymbol{\theta}^1, ..., \boldsymbol{\theta}^M)\ d\mu^1(\boldsymbol{\theta}^1) \dots d\mu^M(\boldsymbol{\theta}^M)\ .$$

The next theorem provides a recursion for $q_n(\boldsymbol{\theta}^1, ..., \boldsymbol{\theta}^M)$.

Theorem 1. Suppose $h\,(\boldsymbol{\theta}^1, ..., \boldsymbol{\theta}^M)$ is the prior density for $\boldsymbol{\alpha}^1, ..., \boldsymbol{\alpha}^M$. Then $q_0(\boldsymbol{\theta}^1, ..., \boldsymbol{\theta}^M) = h\,(\boldsymbol{\theta}^1, ..., \boldsymbol{\theta}^M)$, and the updated estimates are given recursively by

$$q_n(\boldsymbol{\theta}^1, ..., \boldsymbol{\theta}^M) =$$

$$= \sum_{i=1}^{M} \sum_{\ell^i \in \boldsymbol{S}_{X^i_{n-1}}} \frac{\phi_n\left(\sum_{i=1}^{M} \boldsymbol{\ell}^i + \boldsymbol{U}_n - \boldsymbol{X}_n\right)}{\psi(\boldsymbol{X}_n)} \prod_{i=1}^{M} \binom{X^i_{n-1}}{\ell^i_i \ell^i_{i+1} \dots \ell^i_{M+1}} \times$$

$$\times (\theta^i_i)^{\ell^i_i} (\theta^i_{i+1})^{\ell^i_{i+1}} \dots (\theta^i_{M+1})^{\ell^i_{M+1}}\ f\,(\boldsymbol{\theta}^1, ..., \boldsymbol{\theta}^M)\ q_{n-1}\,(\boldsymbol{\theta}^1, ..., \boldsymbol{\theta}^M).$$

Here

$$\boldsymbol{S}_{X^1_{n-1}} = \{\boldsymbol{\ell}^1 = (\ell^1_1, \ell^1_2, ..., \ell^1_{M+1})' : \sum_{i=1}^{M+1} \ell^1_i = X^1_{n-1}\},$$

$$\boldsymbol{S}_{X^2_{n-1}} = \{\boldsymbol{\ell}^2 = (0, \ell^2_2, \ell^2_3, ..., \ell^2_{M+1})' : \sum_{i=2}^{M+1} \ell^2_i = X^2_{n-1}\},$$

$$\dots$$

$$\boldsymbol{S}_{X^M_{n-1}} = \{\boldsymbol{\ell}^M = (0, 0, ..., \ell^M_M, \ell^M_{M+1})' : \ell^M_M + \ell^M_{M+1} = X^M_{n-1}\},$$

$$\binom{X^i_{n-1}}{\ell^i_1 \ell^i_2 \dots \ell^i_{M+1} M} = \frac{X^i_{n-1}!}{\ell^i_1! \ell^i_2! \dots \ell^i_{M+1}!},$$

$$\boldsymbol{\theta}^1 = \begin{bmatrix} \theta^1_1 \\ \theta^1_2 \\ \vdots \\ \theta^1_{M+1} \end{bmatrix}, \quad \boldsymbol{\theta}^2 = \begin{bmatrix} 0 \\ \theta^2_2 \\ \vdots \\ \theta^2_{M+1} \end{bmatrix}, ..., \quad \boldsymbol{\theta}^M = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \theta^M_M \\ \theta^M_{M+1} \end{bmatrix},$$

$$\boldsymbol{\ell}^1 = \begin{bmatrix} \ell_1^1 \\ \ell_2^1 \\ \vdots \\ \ell_{M+1}^1 \end{bmatrix}, \ \boldsymbol{\ell}^2 = \begin{bmatrix} 0 \\ \ell_2^2 \\ \vdots \\ \ell_{M+1}^2 \end{bmatrix}, \ldots, \ \boldsymbol{\ell}^M = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \ell_M^M \\ \ell_{M+1}^M \end{bmatrix}.$$

P r o o f. Let f be a test function then by definition:

$$\overline{E}\,[\Lambda_n f\,(\boldsymbol{\alpha}^1,\ldots,\boldsymbol{\alpha}^M)\,|\,\mathcal{F}_n =$$

$$= \int_{\Theta^1\times\ldots\times\Theta^M} f_n\,(\boldsymbol{\theta}^1,\ldots,\boldsymbol{\theta}^M)\ q_n\,(\boldsymbol{\theta}^1,\ldots,\boldsymbol{\theta}^M)\ d\mu^1(\boldsymbol{\theta}^1)\ldots d\mu^M(\boldsymbol{\theta}^M)\,.$$

However

$$\overline{E}\,[\Lambda_n f\,(\boldsymbol{\alpha}^1,\ldots,\boldsymbol{\alpha}^M)\,|\mathcal{F}_n =$$

$$= \overline{E}\left[\Lambda_{n-1}\frac{\phi_n\left(\sum_{i=1}^{M}\boldsymbol{\alpha}^i \lozenge \langle \boldsymbol{X}_{n-1}, e_i\rangle + \boldsymbol{U}_n - \boldsymbol{X}_n\right)}{\psi(\boldsymbol{X}_n)}\ f\,(\boldsymbol{\alpha}^1,\ldots,\ \boldsymbol{\alpha}^M)|\mathcal{F}_n\right] =$$

$$= \overline{E}\left[\int_{\Theta^1\times\ldots\times\Theta^M}\frac{\phi_n\left(\sum_{i=1}^{M}\boldsymbol{\theta}^i \lozenge \langle \boldsymbol{X}_{n-1}, e_i\rangle + \boldsymbol{U}_n - \boldsymbol{X}_n\right)}{\psi(\boldsymbol{X}_n)}\ f(\boldsymbol{\theta}^1,\ldots,\boldsymbol{\theta}^M)\right],$$

$$q_{n-1}(\boldsymbol{\theta}^1,\ldots,\boldsymbol{\theta}^M)\times d\mu^1(\boldsymbol{\theta}^1)\ldots d\mu^M(\boldsymbol{\theta}^M)|\mathcal{F}_{n-1} =$$

$$= \sum_{i=1}^{M}\ \sum_{\boldsymbol{\ell}^i\in \boldsymbol{S}_{X_{n-1}^i}} \overline{E}\left[\int_{\Theta^1\times\ldots\times\Theta^M}\frac{\phi_n\left(\sum_{i=1}^{M}\ell^i + \boldsymbol{U}_n - \boldsymbol{X}_n\right)}{\psi(\boldsymbol{X}_n)}\prod_{i=1}^{M} I\,(\boldsymbol{Y}^i = \boldsymbol{\ell}^i),\right.$$

$$\left. f\,(\boldsymbol{\theta}^1,\ldots,\boldsymbol{\theta}^M)\,q_{n-1}(\boldsymbol{\theta}^1,\ldots,\boldsymbol{\theta}^M)\,d\mu^1(\boldsymbol{\theta}^1)\ \ldots\ d\mu^M(\boldsymbol{\theta}^M)|\,\mathcal{F}_{n-1}\right] =$$

$$= \sum_{i=1}^{M}\ \sum_{\boldsymbol{\ell}^i\in \boldsymbol{S}_{X_{n-1}^i}}\ \int_{\Theta^1\times\ldots\times\Theta^M}\frac{\phi_n\left(\sum_{i=1}^{M}\ell^i + \boldsymbol{U}_n - \boldsymbol{X}_n\right)}{\psi(\boldsymbol{X}_n)}\prod_{i=1}^{M}\binom{X_{n-1}^i}{\ell_i^i\ell_{i+1}^i\ldots\ell_{M+1}^i}\times$$

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$$\times(\theta_i^i)^{\ell_i^i}(\theta_{i+1}^i)^{\ell_{i+1}^i}\dots(\theta_{M+1}^i)^{\ell_{M+1}^i} f(\boldsymbol{\theta}^1,\dots,\boldsymbol{\theta}^M)\times$$
$$\times\, q_{n-1}(\boldsymbol{\theta}^1,\dots,\boldsymbol{\theta}^M)\, d\mu^1(\boldsymbol{\theta}^1)\dots d\mu^M(\boldsymbol{\theta}^M).$$

Since f is an arbitrary test function the result follows.

3. A finite state case with Markovian dynamics. In this section we assume that each group of parameters $\boldsymbol{\alpha}^1=(\alpha_1^1,\dots,\alpha_{M+1}^1)$, $\boldsymbol{\alpha}^2=(\alpha_2^2,\dots,\alpha_{M+1}^2)$, ..., $\boldsymbol{\alpha}^M=$ $=(\alpha_M^M,\alpha_{M+1}^M)$ is a set of dependent finite-state Markov chains. For the sake of simplicity we suppose that the Markov chains in group i have state spaces equal to K^i. More precisely we have for $n\ge 0$:

$$\alpha_1^1(n)\in\{p_{11}^1,\dots,p_{11}^{K^1}\},\dots,\alpha_{M+1}^1(n)\in\{p_{1(M+1)}^1,\dots,p_{1(M+1)}^{K^1}\},$$
$$\alpha_2^2(n)\in\{p_{22}^1,\dots,p_{22}^{K^2}\},\dots,\alpha_{M+1}^2(n)\in\{p_{2(M+1)}^1,\dots,p_{2(M+1)}^{K^2}\},$$
$$\dots$$
$$\alpha_M^M(n)\in\{p_{MM}^1,\dots,p_{MM}^{K^M}\},\alpha_{M+1}^M(n)\in\{p_{M(M+1)}^1,\dots,p_{M(M+1)}^{K^M}\}.$$

Without any loss of generality, we identify the state space of each Markov chain in group i, with the set of standard unit vectors $\mathbf{R}^{K^i}$.

Write $\mathcal{F}_n$ for the complete filtration generated by the observed inventory $\boldsymbol{X}$, and $\mathcal{G}_n$ for the complete filtration generated by the inventory and the processes $\boldsymbol{\alpha}^1=(\alpha_1^1,\dots,\alpha_{M+1}^1)$, $\boldsymbol{\alpha}^2=(\alpha_2^2,\dots,\alpha_{M+1}^2)$, ..., $\boldsymbol{\alpha}^M=(\alpha_M^M,\alpha_{M+1}^M)$.

Now we define the probability transitions of the above processes. With $\otimes$ denoting the tensor product of two vectors we assume the following:

$$P\left[\bigotimes_{\ell=1}^{M+1}\alpha_\ell^1(n)=\bigotimes_{\ell=1}^{M+1}e_{s_\ell}^1\,|\,\mathcal{G}_{n-1}\right]=$$
$$=P\left[\bigotimes_{\ell=1}^{M+1}\alpha_\ell^1(n)=\bigotimes_{\ell=1}^{M+1}e_{s_\ell}^1\,\middle|\,\alpha_1^1(n-1),\dots,\alpha_{M+1}^1(n-1)\right],$$
$$P\left[\bigotimes_{\ell=2}^{M+1}\alpha_\ell^2(n)=\bigotimes_{\ell=2}^{M+1}e_{s_\ell}^2\,|\,\mathcal{G}_{n-1}\right]=$$
$$=P\left[\bigotimes_{\ell=2}^{M+1}\alpha_\ell^2(n)=\bigotimes_{\ell=2}^{M+1}e_{s_\ell}^2\,\middle|\,\alpha_2^2(n-1),\dots,\alpha_{M+1}^2(n-1)\right],$$
$$\dots$$
$$P\left[\alpha_M^M(n)\otimes\alpha_{M+1}^M(n)=e_{s_M}^M\otimes e_{s_{M+1}}^M\,|\,\mathcal{G}_{n-1}]\right]=$$

$$= P\left[\alpha_M^M(n) \otimes \alpha_{M+1}^M(n) = e_{s_M}^M \otimes e_{s_{M+1}}^M \middle| \alpha_M^M(n-1), \ldots, \alpha_{M+1}^M(n-1) \right].$$

Write

$$P\left[\underset{\ell=1}{\overset{M+1}{\otimes}} \alpha_\ell^1(n) = \underset{\ell=1}{\overset{M+1}{\otimes}} e_{s_\ell}^1 \middle| \alpha_1^1(n-1) = e_{r_1}^1, \ldots, \alpha_{M+1}^1(n-1) = e_{r_{M+1}}^1 \right] =$$

$$= a^1_{s_1^1, \ldots, s_{M+1}^1; r_1^1, \ldots, r_{M+1}^1},$$

$$P\left[\underset{\ell=2}{\overset{M+1}{\otimes}} \alpha_\ell^2(n) = \underset{\ell=2}{\overset{M+1}{\otimes}} e_{s_\ell}^2 \middle| \alpha_2^2(n-1) = e_{r_2}^2, \ldots, \alpha_{M+1}^2(n-1) = e_{r_{M+1}}^2 \right] =$$

$$= a^2_{s_2^2, \ldots, s_{M+1}^2; r_2^2, \ldots, r_{M+1}^2},$$

$$\ldots$$

$$P\left[\alpha_M^M(n) \otimes \alpha_{M+1}^M(n) = e_{s_M}^M \otimes e_{s_{M+1}}^M \middle| \alpha_M^M(n-1) = e_{r_M}^M, \alpha_{M+1}^M(n-1) = e_{r_{M+1}}^M \right] =$$

$$= a^M_{s_M^M, \ldots, s_{M+1}^M; r_M^M, \ldots, r_{M+1}^M};$$

$$A^1 = \{a^1_{s_1^1, \ldots, s_{M+1}^1; r_1^1, \ldots, r_{M+1}^1}\},\ s_1^1, \ldots, s_{M+1}^1,\ r_1^1, \ldots, r_{M+1}^1 = 1, 2, \ldots, K^1,$$

$$A^2 = \{a^2_{s_2^2, \ldots, s_{M+1}^2; r_2^2, \ldots, r_{M+1}^2}\},\ s_2^2, \ldots, s_{M+1}^2,\ r_2^2, \ldots, r_{M+1}^2 = 1, 2, \ldots, K^2,$$

$$\ldots$$

$$A^M = \{a^M_{s_M^M, \ldots, s_{M+1}^M; r_M^M, \ldots, r_{M+1}^M}\},\ s_M^M, \ldots, s_{M+1}^M,\ r_M^M, r_{M+1}^M = 1, 2, \ldots, K^M.$$

We have the following representations [12, 13]:

$$\underset{i=1}{\overset{M+1}{\otimes}} \alpha_i^1(n) = A^1 \underset{i=1}{\overset{M+1}{\otimes}} \alpha_i^1(n-1) + W_n^1,$$

$$\underset{i=2}{\overset{M+1}{\otimes}} \alpha_i^2(n) = A^2 \underset{i=2}{\overset{M+1}{\otimes}} \alpha_i^2(n-1) + W_n^2, \ldots \tag{5}$$

$$\ldots, \alpha_M^M(n) \otimes \alpha_{M+1}^M(n) = A^M \alpha_M^M(n-1) \otimes \alpha_{M+1}^M(n-1) + W_n^M.$$

Here W^1 is a martingale increment process with respect to the filtration generated by the processes $\alpha_1^1, \ldots, \alpha_{M+1}^1$, W^2 is a martingale increment process with respect to the filtration generated by the processes $\alpha_2^2, \ldots, \alpha_{M+1}^2$, W^M is a martingale increment process with respect to the filtration generated by the processes $\alpha_M^M, \alpha_{M+1}^M$. Since

$$\left[\underset{\ell=1}{\overset{M+1}{\otimes}} \alpha_\ell^1(n) = \underset{\ell=1}{\overset{M+1}{\otimes}} e_{s_\ell}^1\right] \Leftrightarrow \left[\alpha_1^1(n) = p_{11}^{s^{11}}, \dots, \alpha_{M+1}^1(n) = p_{1(M+1)}^{s^{1(M+1)}}\right],$$

$$\left[\underset{\ell=2}{\overset{M+1}{\otimes}} \alpha_\ell^2(n) = \underset{\ell=2}{\overset{M+1}{\otimes}} e_{s_\ell}^2\right] e \Leftrightarrow \left[\alpha_2^2(n) = p_{22}^{s^{22}}, \dots, \alpha_{M+1}^2(n) = p_{2(M+1)}^{s^{2(M+1)}}\right],$$

$$\dots$$

$$\left[\alpha_M^M(n) \otimes \alpha_{M+1}^M(n) = e_{s_M}^M \otimes e_{s_{M+1}}^M\right] \Leftrightarrow \left[\alpha_M^M(n) = p_{MM}^{s^{MM}}, \alpha_{M+1}^M(n) = p_{M(M+1)}^{s^{M(M+1)}}\right].$$

Therefore

$$\boldsymbol{\alpha}_n^1 \lozenge \langle \boldsymbol{X}_{n-1}, e_1 \rangle = \sum_{s^{11},\dots,s^{1(M+1)}=1}^{K^1} \begin{bmatrix} p_{11}^{s^{11}} \\ p_{12}^{s^{12}} \\ \vdots \\ p_{1(M+1)}^{s^{1(M+1)}} \end{bmatrix} \lozenge \langle \boldsymbol{X}_{n-1}, e_1 \rangle \times$$

$$\times I\,[\alpha_1^1(n) = p_{11}^{s^{11}}, \dots, \alpha_{M+1}^1(n) = p_{1(M+1)}^{s^{1(M+1)}}],$$

$$\boldsymbol{\alpha}_n^2 \lozenge \langle \boldsymbol{X}_{n-1}, e_2 \rangle = \sum_{s^{22},\dots,s^{2(M+1)}=1}^{K^2} \begin{bmatrix} 0 \\ p_{22}^{s^{22}} \\ \vdots \\ p_{2(M+1)}^{s^{2(M+1)}} \end{bmatrix} \lozenge \langle \boldsymbol{X}_{n-1}, e_2 \rangle \times$$

$$\times I\,[\alpha_2^2(n) = p_{22}^{s^{22}}, \dots, \alpha_{M+1}^2(n) = p_{2(M+1)}^{s^{2(M+1)}}],$$

$$\boldsymbol{\alpha}_n^M \lozenge \langle \boldsymbol{X}_{n-1}, e_M \rangle = \sum_{s^{MM},\dots,s^{M(M+1)}=1}^{K^M} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ p_{MM}^{s^{MM}} \\ p_{M(M+1)}^{s^{M(M+1)}} \end{bmatrix} \lozenge \langle \boldsymbol{X}_{n-1}, e_M \rangle \times$$

$$\times I\,[\alpha_M^M(n) = p_{MM}^{s^{MM}}, \alpha_{M+1}^M(n) = p_{M(M+1)}^{s^{M(M+1)}}],$$

and the dynamics in (4) take the form

$$\boldsymbol{X}_n = \sum_{i=1}^{M} \boldsymbol{\alpha}_n^i \lozenge \langle \boldsymbol{X}_{n-1}, e_i \rangle + \boldsymbol{U}_n - \boldsymbol{V}_n. \tag{6}$$

Write

$$\boldsymbol{p}^1 = \begin{bmatrix} p_{11}^{s^{11}} \\ p_{12}^{s^{12}} \\ \vdots \\ p_{1(M+1)}^{s^{1(M+1)}} \end{bmatrix}, \boldsymbol{p}^2 = \begin{bmatrix} 0 \\ p_{22}^{s^{22}} \\ \vdots \\ p_{2(M+1)}^{s^{2(M+1)}} \end{bmatrix}, \ldots, \boldsymbol{p}^M = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ p_{MM}^{s^{MM}} \\ p_{M(M+1)}^{s^{M(M+1)}} \end{bmatrix}$$

We shall adopt a similar approach to that used in the previous section and work under a reference probability measure $\overline{P}$ where the process $\boldsymbol{X}$ is a sequence of i.i.d. random variables with probability distribution ψ. Set $\lambda_0 = 1$, and $\Lambda_n = \prod_{k=0}^{n} \lambda_k$, with

$$\lambda_k = \frac{\phi_n \left(\sum_{i=1}^{M} \boldsymbol{\alpha}_k^i \lozenge \langle \boldsymbol{X}_{k-1}, e_i \rangle + \boldsymbol{U}_k - \boldsymbol{X}_k \right)}{\psi(\boldsymbol{X}_k)}, \quad k = 1, \ldots .$$

It can be shown that the process $\{\Lambda_n\}$ is a martingale with respect to the filtration $\mathcal{G}_n$. Therefore, we can relate P and $\overline{P}$ by setting $\frac{dP}{d\overline{P}}\Big|_n \triangleq \Lambda_n$.

The probability measure P is the «real world» measure. Using similar arguments to those used in [12], we can also show that under the «real world» measure P that the dynamics in (6) hold. We shall be interested in computing

$$E\left[\left\langle \bigotimes_{\ell=1}^{M+1} \alpha_\ell^1(n), \bigotimes_{\ell=1}^{M+1} e_{s_\ell}^1 \right\rangle \left\langle \bigotimes_{\ell=2}^{M+1} \alpha_\ell^2(n), \bigotimes_{\ell=2}^{M+1} e_{s_\ell}^2 \right\rangle \ldots \left\langle \alpha_M^M(n) \otimes \alpha_{M+1}^M(n), e_{s_M}^M \otimes e_{s_{M+1}}^M \right\rangle \Big| \mathcal{F}_n \right]. \tag{7}$$

A generalized version of Bayes Theorem: see [12] shows that equation (7) is equal to:

$$\frac{\overline{E}[\Lambda_n \left\langle \bigotimes_{\ell=1}^{M+1} \alpha_\ell^1(n), \bigotimes_{\ell=1}^{M+1} e_{s_\ell}^1 \right\rangle \left\langle \bigotimes_{\ell=2}^{M+1} \alpha_\ell^2(n), \bigotimes_{\ell=2}^{M+1} e_{s_\ell}^2 \right\rangle \ldots \left\langle \alpha_M^M(n) \otimes \alpha_{M+1}^M(n), e_{s_M}^M \otimes e_{s_{M+1}}^M \right\rangle | \mathcal{F}_n]}{\overline{E}[\Lambda_n | \mathcal{F}_n]}.$$

The numerator of the above expression represents a unnormalized conditional expectation. Let this expression be denoted by $q_n(\boldsymbol{s}^1, \boldsymbol{s}^2, \ldots, \boldsymbol{s}^M)$, where

$$\boldsymbol{s}^1 = (s_1^1, \ldots, s_{M+1}^1), \boldsymbol{s}^2 = (s_2^2, \ldots, s_{M+1}^2), \ldots, \boldsymbol{s}^M = (s_M^M, s_{M+1}^M).$$

The next theorem gives a recursion for $q_n(\boldsymbol{s}^1, \boldsymbol{s}^2, \ldots, \boldsymbol{s}^M)$.

Theorem 2. Suppose p_0 is the probability distribution of $\alpha^1, ..., \alpha^M$. Then for $n \geq 1$ the updated estimates are given recursively by

$$q_n(\boldsymbol{s}^1, \boldsymbol{s}^2, ..., \boldsymbol{s}^M) =$$

$$= \sum_{r_1^1, ..., r_{M+1}^1=1}^{K_1} \sum_{r_2^2, ..., r_{M+1}^2=1}^{K_2} ... \sum_{r_M^M, r_{M+1}^M=1}^{K_M} a^1_{s_1^1, ..., s_{M+1}^1; r_1^1, ..., r_{M+1}^1} a^2_{s_2^2, ..., s_{M+1}^2; r_2^2, ..., r_{M+1}^2} ...$$

$$... a^M_{s_M^M, ..., s_{M+1}^M; r_M^M, ..., r_{M+1}^M} \sum_{i=1}^{M} \sum_{\boldsymbol{\ell}^i \in \boldsymbol{S}_{X_{n-1}^i}} \frac{\phi_n\left(\sum_{i=1}^{m} \boldsymbol{\ell}^i + \boldsymbol{U}_n - \boldsymbol{X}_n\right)}{\psi(\boldsymbol{X}_n)} \prod_{i=1}^{m} \binom{X_{n-1}^i}{\ell_i^i \ell_{i+1}^i ... \ell_{M+1}^i} \times$$

$$\times (p_{ii}^{s^{ii}})^{\ell_i^i} (p_{i(i+1)}^{s^{i(i+1)}})^{\ell_{i+1}^i} ... (p_{i(M+1)}^{s^{i(M+1)}})^{\ell_{M+1}^i} q_{n-1}(\boldsymbol{r}^1, ..., \boldsymbol{r}^M).$$

Here

$$\boldsymbol{r}^1 = (r_1^1, ..., r_{M+1}^1),\ \boldsymbol{r}^2 = (r_2^2, ..., r_{M+1}^2),\ ...,\ \boldsymbol{r}^M = (r_M^M, r_{M+1}^M).$$

P r o o f. First note that

$$\overline{E}\left[\Lambda_n \left\langle \bigotimes_{\ell=1}^{M+1} \alpha_\ell^1(n), \bigotimes_{\ell=1}^{M+1} e_{s_\ell}^1 \right\rangle \left\langle \bigotimes_{\ell=2}^{M+1} \alpha_\ell^2(n), \bigotimes_{\ell=2}^{M+1} e_{s_\ell}^2 \right\rangle ... \left\langle \alpha_M^M(n) \otimes \alpha_{M+1}^M(n), e_{s_M}^M \otimes \right.\right.$$

$$\left.\left. \otimes\, e_{s_{M+1}}^M \right\rangle | \mathcal{F}_n \right] = \overline{E}\,[\Lambda_{n-1} \left\langle \bigotimes_{\ell=1}^{M+1} \alpha_\ell^1(n), \bigotimes_{\ell=1}^{M+1} e_{s_\ell}^1 \right\rangle \left\langle \bigotimes_{\ell=2}^{M+1} \alpha_\ell^2(n), \bigotimes_{\ell=2}^{M+1} e_{s_\ell}^2 \right\rangle ...$$

$$... \left\langle \alpha_M^M(n) \otimes \alpha_{M+1}^M(n), e_{s_M}^M \otimes e_{s_{M+1}}^M \right\rangle.$$

Using the representations in (5) this is

$$\frac{\phi_n\left(\sum_{i=1}^{M} \boldsymbol{p}^i \Diamond \langle \boldsymbol{X}_{n-1}, e_i \rangle + \boldsymbol{U}_n - \boldsymbol{X}_n\right)}{\psi(\boldsymbol{X}_n)} | \mathcal{F}_n] =$$

$$= \overline{E}\left[\Lambda_{n-1} \left\langle A^1 \bigotimes_{i=1}^{M+1} \alpha_i^1(n-1), \bigotimes_{\ell=1}^{M+1} e_{s_\ell}^1 \right\rangle \left\langle A^2 \bigotimes_{i=2}^{M+1} \alpha_i^2(n-1), \bigotimes_{\ell=2}^{M+1} e_{s_\ell}^2 \right\rangle ... \right.$$

$$... \left\langle A^M \alpha_M^M(n-1) \otimes \alpha_{M+1}^M(n-1), e_{s_M}^M \otimes e_{s_{M+1}}^M \right\rangle,$$

$$\left. \frac{\phi_n\left(\sum_{i=1}^{M} \boldsymbol{p}^i \Diamond \langle \boldsymbol{X}_{n-1}, e_i \rangle + \boldsymbol{U}_n - \boldsymbol{X}_n\right)}{\psi(\boldsymbol{X}_n)} | \mathcal{F}_n \right] =$$

$$= \sum_{r_1^1, \ldots, r_{M+1}^1 = 1}^{K_1} \sum_{r_2^2, \ldots, r_{M+1}^2 = 1}^{K_2} \ldots \sum_{r_M^M, r_{M+1}^M = 1}^{K_M} a^1_{s_1^1, \ldots, s_{M+1}^1; r_1^1, \ldots, r_{M+1}^1} a^2_{s_2^2, \ldots, s_{M+1}^2; r_2^2, \ldots, r_{M+1}^2} \ldots$$

$$\ldots a^M_{s_M^M, \ldots, s_{M+1}^M; r_M^M, \ldots, r_{M+1}^M} \overline{E}[\Lambda_{n-1} \left\langle \bigotimes_{i=1}^{M+1} \alpha_i^1(n-1), \bigotimes_{\ell=1}^{M+1} e^1_{r_\ell} \right\rangle \left\langle \bigotimes_{i=2}^{M+1} \alpha_i^2(n-1), \bigotimes_{\ell=2}^{M+1} e^2_{r_\ell} \right\rangle \ldots$$

$$\ldots \left\langle \alpha_M^M(n-1) \otimes \alpha_{M+1}^M(n-1), e^M_{r_M} \otimes e^M_{r_{M+1}} \right\rangle,$$

$$\frac{\phi_n\left(\sum_{i=1}^{M} \boldsymbol{p}^i \lozenge \langle \boldsymbol{X}_{n-1}, e_i \rangle + \boldsymbol{U}_n - \boldsymbol{X}_n \right)}{\psi(\boldsymbol{X}_n)} | \mathcal{F}_{n-1}] =$$

$$= \sum_{r_1^1, \ldots, r_{M+1}^1 = 1}^{K_1} \sum_{r_2^2, \ldots, r_{M+1}^2 = 1}^{K_2} \ldots \sum_{r_M^M, r_{M+1}^M = 1}^{K_M} a^1_{s_1^1, \ldots, s_{M+1}^1; r_1^1, \ldots, r_{M+1}^1} a^2_{s_2^2, \ldots, s_{M+1}^2; r_2^2, \ldots, r_{M+1}^2} \ldots$$

$$\ldots a^2_{s_M^M, \ldots, s_{M+1}^M; r_M^M, \ldots, r_{M+1}^2} \sum_{i=1}^{M} \sum_{\boldsymbol{\ell}^i \in \boldsymbol{S}_{X_{n-1}^i}} \frac{\phi_n\left(\sum_{i=1}^{m} \boldsymbol{\ell}^i + \boldsymbol{U}_n - X_n \right)}{\psi(\boldsymbol{X}_n)} \prod_{i=1}^{m} \binom{X_{n-1}^i}{\ell_i^i \ell_{i+1}^i \ldots \ell_{M+1}^i} \times$$

$$\times (p_{ii}^{s^{ii}})^{\ell_i^i} (p_{i(i+1)}^{s^{i(i+1)}})^{\ell_{i+1}^i} \ldots (p_{i(M+1)}^{s^{i(M+1)}})^{\ell_{M+1}^i} q_{n-1}(\boldsymbol{r}^1, \ldots, \boldsymbol{r}^M).$$

Here

$$\boldsymbol{r}^1 = (r_1^1, \ldots, r_{M+1}^1), \ \boldsymbol{r}^2 = (r_2^2, \ldots, r_{M+1}^2), \ \ldots, \ \boldsymbol{r}^M = (r_M^M, r_{M+1}^M).$$

4. A partially observed inventory. In this section we assume that the inventory level is not observed at all time. However, the management observes the event when the inventory falls to zero and cannot observe the inventory when it is positive. To study such partial observations of the inventory levels, we introduce a signal (message) random variable

$$Z_n^i \stackrel{\Delta}{=} I(X_n^i = 0), \ n = 0, 1, 2, \ldots .$$

The processes Z^i, $i = 1, \ldots, M$ are discrete-time Markov Chains with the state space the set $\{0,1\}$ where 1 means an empty inventory and 0 means a nonempty one. Write

$$\mathcal{G}_n = \sigma\{\boldsymbol{X}_k, Z_k^i, \boldsymbol{\alpha}^i, i = 1, \ldots, M, \boldsymbol{U}_k, \boldsymbol{V}_k, k \le n\},$$

$$\mathcal{F}_n = \sigma\{Z_k^i, i = 1, \ldots, M, U_k, V_k, k \le n\}.$$

We shall suppose that:

$$P[Z_n^i = m|_{n-1}] = P[Z_n^i = m | Z_{n-1}^i, X_{n-1}^i, U_{n-1}, V_{n-1}].$$

Write

$$a_{m,\ell}(X_{n-1}, U_{n-1}, V_{n-1}) = P[Z_n^i = m \mid Z_{n-1}^i = \ell, X_{n-1}, U_{n-1}, V_{n-1}].$$

We shall work under a reference probability measure $\overline{P}$ where the the process $\boldsymbol{X}$ is a sequence of i.i.d. random variables with probability distribution ψ, and the processes Z^i are i.i.d. random variables uniformly distributed on the set $\{0, 1\}$. Set $\lambda_0 = 1$, and for $k \geq 1$,

$$\lambda_k = \frac{\phi_n\left(\sum_{i=1}^{M} \boldsymbol{\alpha}^i \lozenge \langle \boldsymbol{X}_{k-1}, e_i \rangle + \boldsymbol{U}_k - \boldsymbol{X}_k\right)}{\psi(\boldsymbol{X}_k)} \times$$

$$\times \prod_{i=1}^{M} \prod_{\ell, m=0,1} (2a_{m,\ell}^i(\boldsymbol{X}_{k-1}, \boldsymbol{U}_{k-1}, \boldsymbol{V}_{k-1}))^{I(Z_k^i = m, Z_{k-1}^i = \ell)},$$

and $\Lambda_n = \prod_{k=0}^{n} \lambda_k$. It can be shown that the process $\{\Lambda_n\}$ is a martingale with respect to the filtration $\mathcal{G}_n$. Therefore, we can relate P and $\overline{P}$ by setting $\frac{dP}{d\overline{P}}\Big|_{\mathcal{G}_n} \overset{\Delta}{=} \Lambda_n$. Under the «real world» measure P the dynamics in (6) hold. We wish to find a recursion for

$$E\left[\Lambda_n I(\boldsymbol{X}_n = \boldsymbol{x}) \prod_{i=1}^{M} I(\boldsymbol{\alpha}^i \in d\boldsymbol{\theta}^i) \mid \mathcal{F}_n\right].$$

Let this expression be denoted by $q_n(\boldsymbol{s}^1, \boldsymbol{s}^2, \ldots, \boldsymbol{s}^M)$, where

$$\boldsymbol{s}^1 = (s_1^1, \ldots, s_{M+1}^1), \boldsymbol{s}^2 = (s_2^2, \ldots, s_{M+1}^2), \ldots, \boldsymbol{s}^M = (s_M^M, s_{M+1}^M).$$

Theorem 3. Suppose $h(\boldsymbol{x}, \boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^M)$ is the prior density for $\boldsymbol{X}_0$, $\alpha^1, \ldots$ $\ldots, \alpha^M$. Then $q_0(x, \boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^M) = h(x, \boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^M)$, and the updated estimates are given recursively by

$$q_n(\boldsymbol{x}, \boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^M) = \sum_z \prod_{i=1}^{M} \prod_{\ell, m=0,1} (2a_{m,\ell}^i(\boldsymbol{z}, \boldsymbol{U}_{n-1}, \boldsymbol{V}_{n-1}))^{I(Z_n^i = m, Z_{n-1}^i = \ell)} \times$$

$$\times \sum_{i=1}^{M} \sum_{\ell^i \in \boldsymbol{S}_{z^i}} \phi_n\left(\sum_{i=1}^{M} \boldsymbol{\ell}^i + \boldsymbol{U}_n - \boldsymbol{x}\right) \prod_{i=1}^{M} \binom{\boldsymbol{z}}{\ell_i^i \ell_{i+1}^i \ldots \ell_{M+1}^i} \times$$

$$\times (\theta_i^i)^{\ell_i^i} (\theta_{i+1}^i)^{\ell_{i+1}^i} \ldots (\theta_{M+1}^i)^{\ell_{M+1}^i} q_{n-1}(\boldsymbol{z}, \boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^M).$$

For the notation see Theorem 1.

P r o o f. Let f be a test function then by definition:

$$\overline{E}\,[\Lambda_n I(\boldsymbol{X}_n=\boldsymbol{x})\, f\,(\boldsymbol{\alpha}^1,\dots,\boldsymbol{\alpha}^M)\,|\,\mathcal{F}_n=$$

$$=\int_{\Theta^1\times\dots\times\Theta^M} f(\boldsymbol{\theta}^1,\dots,\boldsymbol{\theta}^M)\; q_n(\boldsymbol{x},\boldsymbol{\theta}^1,\dots,\boldsymbol{\theta}^M)\; d\mu^1(\boldsymbol{\theta}^1)\dots d\mu^M(\boldsymbol{\theta}^M)\quad .$$

However

$$\overline{E}[\Lambda_n I(\boldsymbol{X}_n=\boldsymbol{x})\; f(\boldsymbol{\alpha}^1,\dots,\boldsymbol{\alpha}^M)\,|\,\mathcal{F}_n=$$

$$=\overline{E}\left[I(\boldsymbol{X}_n=\boldsymbol{x})\,\Lambda_{n-1}\frac{\phi_n\left(\sum_{i=1}^M \boldsymbol{\alpha}^i\,\Diamond\,\langle \boldsymbol{X}_{n-1},e_i\rangle+\boldsymbol{U}_n-\boldsymbol{x}\right)}{\psi(\boldsymbol{x})}\times\right.$$

$$\left.\times\prod_{i=1}^M\prod_{\ell,m=0,1}(2a^i_{m,\ell}(\boldsymbol{X}_{n-1},\boldsymbol{U}_{n-1},\boldsymbol{V}_{n-1}))^{I(Z^i_n=m,Z^i_{n-1}=\ell)}\; f(\alpha^1,\dots,\alpha^M)|\mathcal{F}_n\right]=$$

$$=\overline{E}\left[\Lambda_{n-1}\phi_n\left(\sum_{i=1}^M \boldsymbol{\alpha}^i\Diamond\langle \boldsymbol{X}_{n-1},e_i\rangle+\boldsymbol{U}_n-\boldsymbol{x}\right)\times\right.$$

$$\left.\times\prod_{i=1}^M\prod_{\ell,m=0,1}(2a^i_{m,\ell}(\boldsymbol{X}_{n-1},\boldsymbol{U}_{n-1},\boldsymbol{V}_{n-1}))^{I(Z^i_n=m,Z^i_{n-1}=\ell)}\; f\,(\alpha^1,\dots,\alpha^M)|\mathcal{F}_n\right]=$$

$$=\sum_{z}\prod_{i=1}^M\prod_{\ell,m=0,1}(2a^i_{m,\ell}(\boldsymbol{z},\boldsymbol{U}_{n-1},\boldsymbol{V}_{n-1}))^{I(Z^i_n=m,Z^i_{n-1}=\ell)}\times$$

$$\times\overline{E}\,[I(\boldsymbol{X}_{n-1}=\boldsymbol{z})\,\Lambda_{n-1}\phi_n\left(\sum_{i=1}^M \alpha^i\Diamond\langle \boldsymbol{z},e_i\rangle+\boldsymbol{U}_n-\boldsymbol{x}\right)\; f(\alpha^1,\dots,\alpha^M)|\mathcal{F}_n]=$$

$$=\sum_{z}\prod_{i=1}^M\prod_{\ell,m=0,1}(2a^i_{m,\ell}(\boldsymbol{z},\boldsymbol{U}_{n-1},\boldsymbol{V}_{n-1}))^{I(Z^i_n=m,Z^i_{n-1}=\ell)}\times$$

$$\times\overline{E}\left[\int_{\Theta^1\times\dots\times\Theta^M}\phi_n\left(\sum_{i=1}^M \boldsymbol{\theta}^i\Diamond\langle \boldsymbol{z},e_i\rangle+\boldsymbol{U}_n-\boldsymbol{x}\right) f(\boldsymbol{\theta}^1,\dots,\boldsymbol{\theta}^M)\times\right.$$

$$\left.\times\, q_{n-1}(\boldsymbol{z},\boldsymbol{\theta}^1,\dots,\boldsymbol{\theta}^M)\,d\mu^1(\boldsymbol{\theta}^1)\dots d\mu^M(\boldsymbol{\theta}^M)\,|\mathcal{F}_{n-1}\right]=$$

$$=\sum_{z}\prod_{i=1}^M\prod_{\ell,m=0,1}(2a^i_{m,\ell}(\boldsymbol{z},\boldsymbol{U}_{n-1},\boldsymbol{V}_{n-1}))^{I(Z^i_n=m,Z^i_{n-1}=\ell)}\times$$

$$\times \sum_{i=1}^{M} \sum_{\boldsymbol{\ell}^i \in \boldsymbol{S}_{\boldsymbol{X}^i_{n-1}}} \int_{\Theta^1 \times \ldots \times \Theta^M} \phi_n\left(\sum_{i=1}^{M} \boldsymbol{\ell}^i + \boldsymbol{U}_n - \boldsymbol{x}\right) \prod_{i=1}^{M} \binom{\boldsymbol{z}}{\ell^i_i \ell^i_{i+1} \ldots \ell^i_{M+1}} \times$$

$$\times (\theta^i_i)^{\ell^i_i} (\theta^i_{i+1})^{\ell^i_{i+1}} \ldots (\theta^i_{M+1})^{\ell^i_{M+1}} f(\boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^M)\, q_{n-1}(\boldsymbol{z}, \boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^M) \times$$

$$\times\, d\mu^1(\boldsymbol{\theta}^1) \ldots d\mu^M(\boldsymbol{\theta}^M).$$

Since f is an arbitrary test function the result follows.

In this paper we proposed a multivariate discrete-time, discrete state stochastic inventory model for perishable items. The proposed model is based on the multinomial thinning used in integer-valued time series analysis and for modeling count data. The present paper was concerned with estimating the vector valued parameter process of the multinomial thinning where recursive estimators were proposed from the Bayesian point of view, the dynamic view and finally the partial observed case.

Розглянуто стохастичну модель управління запасами з багатьма випадковими змінними, дискретну у часі та просторі, для товарів, що швидко псуються. Модель базується на попередніх роботах авторів з використанням дробового оператора розрідження Стентела та Ван Харна. Прийнято припущення про те, що товари на складі належать до одної з M можливих категорій якості. У кожну мить часу t товари на складі можуть залишатись в одному і тому ж класі, переходити в один із M–1 класів, або псуватись. Припускається також, що пересування поміж класами регулюється мультиноміальним оператором розрідження, який залежить від деякого процесу з параметрами, що векторно оцінюються. Для трьох можливих сценаріїв запропоновано рекурсивні оцінки параметрів процесу.

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