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# Second Method of Lyapunov for Boundedness in Terms of two Measures for Impulsive Functional Differential Systems

This paper studies an initial value problem for impulsive functional differential equations with finite delay and fixed moments of impulse effect. By using piecewise continuous functions coupled with the Razumikhin technique sufficient conditions for boundedness in terms of two different piecewise continuous measures of such equations are found. The results extend and improve the earlier publications.

Исследована задача с начальными условиями для импульсных функциональных дифференциальных уравнений с конечным запаздыванием и фиксированными моментами импульсного воздействия. В результате использования кусочно-непрерывных функций и метода Разумихина найдены достаточные условия ограниченности по двум различным кусочно-непрерывным критериям таких уравнений. Приведенные результаты дополняют и подтверждают полученные ранее.

*K e y w o r d s: boundedness in termss of two measures, Razumikhin technique, impulsive functional differential equations.*

**Introduction.** Impulsive equations have a great deal of applications in physics, biology, medicine and other sciences. In [1—5], stability and boundedness properties of impulsive ordinary differential equations are discussed. There also exists a well-developed qualitative theory of functional differential equations. See, for example, [6—9] and the references cited therein.

The impulsive functional differential equations are a natural generalization of functional differential equations (FDE) without impulses and of impulsive ordinary differential equations (IODE) without delay. They are adequate mathematical models of various real processes and phenomena, characterized by the fact that their state changes by jumps and by the dependence of the process on its history at each moments of time. At the present time the theory of impulsive functional differential equations undergoes rapid development. A large number of criteria on the stability of the solutions of such equations have been derived [10—16]. In order to explore Lagrange stability or the existence of periodic solutions of the impulsive functional differential systems, we need to know not only the stability but also the boundedness of solutions. However, very little is known about the boundedness properties of such systems.

The objective of this paper is to investigate the boundedness in terms of two measures for the impulsive systems of functional differential equations with finite delay and fixed moments of impulse effect. The priorities of this approach are useful and well known in the investigations on the stability and boundedness of the solutions of FDE without impulses, as well as in the generalizations obtained by this method [4, 6, 17].

We give some preliminary results and basic definitions which will be used. Sufficient conditions for boundedness in terms of two different piecewise continuous measures of the impulsive nonlinear functional differential equations under impulsive perturbations at fixed moments of time are given. The investigation is carried by employing a class of piecewise continuous functions which are generalizations of the clasical Lyapunov's functions. Moreover, the technique of investigation essentially depends on the choice of minimal subsets of a suitable space of piecewise continuous functions, by the elements of which the derivatives of Lyapunov's functions are estimated. It is well known that Lyapunov-Razumikhin function method have been widely used in the treatment of the stability of FDE without impulses [3, 6, 7, 18]. Such a method applied to the investigation of various type of stability of impulsive functional differential equations can be found in [10—16].

**Preliminary notes and definitions.** Let $R^n$ be the $n$-dimensional Euclidean space with norm $|\cdot|$; $R_+=[0,\infty)$. Let $r>0$ and $E=\{\phi:[-r,0]\to R^n,\ \phi(t)$ is continuous everywhere except at finite number of points $t=\tau_k\in[-r,0]$ at which $\phi(\tau_k-0)$ and $\phi(\tau_k+0)$ exist and $\phi(\tau_k-0)=\phi(\tau_k)\}$. If $t>t_0$, $t_0\in R_+$ we define $x_t\in E$ by $x_t=x(t+s)$, $-r\le s\le 0.$ Consider the system of impulsive functional differential equations

$$\dot{x}(t)=f(t,x_t),\ t>t_0,\ t\ne\tau_k,$$
$$\Delta x(\tau_k)=x(\tau_k+0)-x(\tau_k-0)=I_k(x(\tau_k-0)),\tau_k>t_0, \tag{1}$$

where $f:[t_0,\infty)\times E\to R^n$; $I_k:R^n\to R^n$; $\tau_k<\tau_{k+1}$ with $\lim\limits_{k\to\infty}\tau_k=\infty.$

Let $\phi\in E.$ Denote by $x(t)=x(t;t_0,\phi),\ x\in R^n$ the solution of system (1) satisfying the initial conditions :

$$x(t;t_0,\phi)=\phi(t-t_0),\ t_0-r\le t\le t_0,$$
$$x(t_0+0;t_0,\phi)=\phi(0)\,. \tag{2}$$

The solution $x(t)=x(t;t_0,\phi)$ of the initial value problem (1),(2) is characterized by the following :

a) for $t_0 - r \le t \le t_0$ the solution $x(t)$ satisfied the initial conditions (2);

b) in the interval $[t_0, \infty)$ the solution $x(t; t_0, \phi)$ of problem (1), (2) is a piecewise continuous function with points of discontinuity of the first kind $t = \tau_k$, $\tau_k \in [t_0, \infty)$ at which it is continuous from the left.

Let $\tau_0 = t_0 - r$. Introduce the following notations:

$$I_0 = [t_0 - r, \infty);\ G_k = \{(t, x) \in I_0 \times R^n : \tau_{k-1} < t < \tau_k\},\ k = 1, 2, \ldots;\ G = \bigcup_{k=1}^{\infty} G_k.$$

**Definition 1.** We shall say that the function $V : I_0 \times R^n \to R_+$ belongs to the class $V_0$ if:

1. The function $V$ is continuous in $G$ and locally Lipschitz continuous with respect to its second argument in each of the sets $G_k$, $k = 1, 2, \ldots$

2. For each $k = 1, 2, \ldots$ and $x \in R^n$ there exist the finite limits

$$V(\tau_k - 0, x) = \lim_{\substack{t \to \tau_k \\ t < \tau_k}} V(t, x),\ V(\tau_k + 0, x) = \lim_{\substack{t \to \tau_k \\ t > \tau_k}} V(t, x).$$

3. The equality $V(\tau_k - 0, x) = V(\tau_k, x)$ is valid.

In the sequel we will use the next classes of functions:

$$K = \{a \in C[R_+, R_+] : a(r) \text{ is strictly increasing and } a(0) = 0\};$$

$$\Gamma = \{h \in V_0 : \inf_{x \in R^n} h(t, x) = 0 \text{ for each } t \in I_0\}.$$

**Definition 2.** Let $h, h^0 \in \Gamma$ and define for $\phi \in E$

$$h_0(t, \phi) = \sup_{-r \le s \le 0} h^0(t + s, \phi(s)),$$
$$\bar{h}(t, \phi) = \sup_{-r \le s \le 0} h(t + s, \phi(s)). \tag{3}$$

We will use the following definitions of boundedness of the system (1) in terms of two different measures.

**Definition 3.** Let $h, h_0 \in \Gamma$. and $h_0$ is defined by (3). The system (1) is said to be:

a) $(h_0, h)$ — uniformly bounded if

$$(\forall \alpha > 0)(\exists \beta = \beta(\alpha) > 0)(\forall t_0 \in R_+),$$
$$h_0(t_0, \phi) < \alpha \text{ implies } h(t, x(t; t_0, \phi)) < \beta,\ t \ge t_0.$$

b) $(h_0, h)$ — quasi uniformly ultimately bounded if

$$(\exists \beta > 0)(\forall \alpha > 0)(\exists T = T(\alpha) > 0)(\forall t_0 \in R_+),$$
$$h_0(t_0, \phi) < \alpha \text{ imples } h(t, x(t; t_0, \phi)) < \beta,\ t \ge t_0 + T.$$

c) $(h_0, h)$ — uniformly ultimately bounded if (a) and (b) hold together.

We will use also the following classes of functions:

$$S^c(h^0,\rho)=\{(t,x)\in I_0\times R^n : h^0(t,x)\geq\rho,\rho>0\};$$

$$S^c(h_0,\rho)=\{(t,\phi)\in[t_0,\infty)\times E : h_0(t,\phi)\geq\rho,\rho>0\};$$

$PC[I_0\times R^n]=\{x: I_0\to R^n : x$ is piecewise continuous with points of discontinuity of the first kind $\tau_k, \tau_k\in I_0$ at which it is continuous from the left$\}$;

$PC^1[[t_0,\infty),R^n]=\{x\in PC[[t_0,\infty),R^n]: x$ is continuously differentiable everywhere except the points $\tau_k,\tau_k\in[t_0,\infty)$ at which it $\dot{x}(\tau_k-0)$ and $\dot{x}(\tau_k+0)$ exist and $\dot{x}(\tau_k-0)=\dot{x}(\tau_k)\}$;

$\Omega_1=\{x\in PC[[t_0,\infty),R^n]: V(s,x(s))\leq L(V(t,x(t))),\ t-r<s\leq t,\ t\geq t_0,\ V\in V_0\}$, where $L(u)$ is continuous on $R_+$, nondecreasing in $u$, and $L(u)>u$ for $u>0$;

$\Omega_0=\{x\in PC[[t_0,\infty),R^n]: V(s,x(s))\leq V(t,x(t)),\ t-r<s\leq t,\ t\geq t_0,\ V\in V_0\}$.

Let $V\in V_0$, $t>t_0-r$, $t\neq\tau_k$, $k=1,2,\ldots$ and $x\in PC[I_0,R^n]$. Introduce the function

$$D\_V(t,x(t))=\lim_{\theta\to 0^-} inf\ \theta^{-1}[V(t+\theta,x(t)+\theta f(t,x_t))-V(t,x(t))].$$

Introduce the following assumptions :

A1. The function $f:[t_0,\infty)\times E\to R^n$ is continuous in $[\tau_{k-1},\tau_k)\times E$ and for every $x_t\in E$, $k=1,2,\ldots$ $f(\tau_k-0,x_t)$ and $f(\tau_k+0,x_t)$ exist and $f(\tau_k-0,x_t)=$ $=f(\tau_k,x_t)$.

A2. $I_k\in C[R^n,R^n]$, $k\in N$.

A3. $t_0-r=\tau_0<\tau_1<\tau_2<\ldots$ and $\lim_{k\to\infty}\tau_k=\infty$.

A4. There exists $\rho_0$, $\rho_0\geq\rho>0$, such that $h^0(\tau_k,x)\geq\rho_0$ implies $h^0(\tau_k+$ $+0,x+I_k(x))\geq\rho$, $k=1,2,\ldots$ .

In the proofs of the main theorems we will use the following comparison results.

**Lemma 1** [15] . Assume the following conditions hold :

1. Assumptions A1 — A3 are valid:

2. $g\in PC[[t_0,\infty)\times R_+,R]$ and $g(t,0)=0$ for $t\in[t_0,\infty)$.

3. $B_k\in C[R_+,R_+]$, $B_k(0)=0$ and $\psi_k(u)=u+B_k(u)$ are nondecreasing with respect to $u$, $k=1,2,\ldots$.

4. The maximal solution $r(t;t_0,u_0)$ of the problem

$$\dot{u}(t)=g(t,u(t)),\ t>t_0,\ t\neq\tau_k,$$

$$u(t_0+0)=u_0\geq 0,$$

$$\Delta u(\tau_k)=B_k(u(\tau_k)),$$

is defined in the interval $[t_0,\infty)$.

5. The function $V \in V_0$ is such that for $t > t_0$ and $x \in \Omega_0$ we have

$$D\_V(t, x(t)) \le g(t, V(t, x(t))),\ t \ne \tau_k,$$

$$V(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k))) \le \psi_k(V(\tau_k),\ x(\tau_k))),$$

and $u_0 \ge V(t_0 + 0, \phi(0))$.

6. For the solution $x(t; t_0, \phi)$ of the system (1) we have $x \in PC[I_0, R^n] \cap \cap PC^1[[t_0, \infty), R^n]$. Then

$$V(t, x(t; t_0, \phi)) \le r(t; t_0, u_0) \text{ for } t \in [t_0, \infty).$$

**Corollary 1.** Let the following conditions hold :

1. Assumptions A1—A3 are met.
2. The function $V \in V_0$, is such that for $t > t_0$ and $x \in \Omega_1$ we have

$$D\_V(t, x(t)) \le 0,\ t \ne \tau_k.$$

$$V(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k))) \le V(\tau_k, x(\tau_k)),$$

3. Condition 6 of Lemma 1 holds. Then

$$V(t, x(t; t_0, \phi)) \le V(t_0 + 0, \phi(0)),\ t \in [t_0, \infty).$$

**Main results. Theorem 1.** Assume the following conditions hold :

1. Assumptions A1—A4 are valid:
2. $h, h^0 \in \Gamma$ and $\bar{h}(t, \phi) \le \varphi(h_0(t, \phi))$ for some $\varphi \in K$ where $h_0, \bar{h}$ are defined by (3).
3. For $\rho > 0$, there exists $V \in V_0$ such that

$$V(t, x) \ge a(h(t, x)) \text{ for } (t, x(t)) \in S^c(h^0, \rho), \tag{4}$$

$$V(t + 0, x) \le b(h_0(t, \phi)) \text{ for } (t, \phi) \in S^c(h_0, \rho), \tag{5}$$

where $a, b \in K$ and $a(r) \to \infty$ as $r \to \infty$.

4. For $t \ge t_0$, $(t, x(t)) \in S^c(h^0, \rho)$ and $x \in \Omega_0$ we have

$$D\_V(t, x(t)) \le 0,\ t \ne \tau_k, \tag{6}$$

$$V(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k))) \le V(\tau_k, x(\tau_k)). \tag{7}$$

Then the system (1) is $(h_0, h)$ — uniformly bounded.

**Proof.** Let $\alpha > \rho_0$ be given. Choose $\beta = \beta(\alpha) > 0$ so that

$$\beta > \max\{\rho_0, \varphi(\alpha), a^{-1}(b(\alpha))\}.$$

Let $t_0 \in R_+$ and $\phi \in E$. Consider the solution $x(t) = x(t; t_0, \phi)$ of (1) with $h_0(t_0, \phi) < \alpha$. By the condition 2 of Theorem 1, we have

$$h(t_0 + 0, \phi(0)) \le \bar{h}(t_0, \phi) \le \varphi(h_0(t_0, \phi)) < \varphi(\alpha) < \beta.$$

We claim that $h(t,x(t))<\beta$, $t\geq t_0$. If it is not true, then there exists some solution $x(t)=x(t;t_0,\phi)$ of (1) with $h_0(t_0,\phi)<\alpha$ and a $t^*>t_0$ such that $\tau_k<t^*\leq$ $\leq\tau_{k+1}$ for some fixed integer $k$ and $h(t^*,x(t^*))\geq\beta$ and $h(t,x(t))<\beta$, $t_0<t\leq\tau_k$. Applying now Corollary 1 for the interval $(t_0,\tau_k]$ we obtain

$$V(t,x(t;t_0,\phi))\leq V(t_0+0,\phi(0)),\ t_0<t\leq\tau_k. \tag{8}$$

Since $h^0(\tau_k,x(\tau_k))\geq\rho_0$, condition A4 shows that

$$h^0(\tau_k+0,x(\tau_k+0))=h^0(\tau_k+0,x(\tau_k)+I_k(x(\tau_k)))\geq\rho,$$

i. e. $(\tau_k+0,x(\tau_k+0))\in S^c(h^0,\rho)$. So the implications (4), (7), (8) and (5) lead to

$$a(h(\tau_k+0,x(\tau_k+0)))\leq V(\tau_k+0,x(\tau_k+0))=V(\tau_k+0,x(\tau_k)+I_k(x(\tau_k)))\leq$$
$$\leq V(\tau_k,x(\tau_k))\leq V(t_0+0,\phi(0))\leq b(h_0(t_0,\phi))<b(\alpha)<a(\beta).$$

Therefore $h(\tau_k+0,x(\tau_k+0))<\beta$. Thus there exist $t_1$, $t_2$, $\tau_k\leq t_1<t_2\leq t^*$ such that

$$h^0(t_1,x(t_1))=\alpha,\ h_0(t_1,x_{t_1})=\alpha,\ h(t_2,x(t_2))=\beta,\ \bar{h}(t_2,x_{t_2})=\beta$$

and

$$(t,x(t))\in\overline{S^c(h^0,\alpha)\cap S(h,\beta)},$$
$$(t,x_t)\in\overline{S^c(h_0,\alpha)\cap S(\bar{h},\beta)},\ t\in[t_1,t_2]. \tag{9}$$

By (5) we have $V(t_1+0,x(t_1))=V(t_1,x(t_1))\leq b(h_0(t_1,x_{t_1})=b(\alpha)<a(\beta)$. We want to show that

$$V(t,x(t))<a(\beta),\ t\in[t_1,t_2]. \tag{10}$$

Suppose that this is not true and let $\xi=\inf\{t_2\geq t>t_1:V(t,x(t))\geq a(\beta)\}$.

Since $V(t,x(t))$ is continuous at $\xi\in(t_1,t_2]$ we see that $V(\xi+\theta,x(\xi+\theta))\geq a(\beta)$ holds which implies that $D\_V(\xi,x(\xi))>0$, which contradicts to (6). Hence (10) holds. On the other hand, using (9) and (4) we have $V(t_2,x(t_2))\geq a(h(t_2,x_{t_2}))=a(\beta)$, which contradicts (10). Thus $h(t,x(t))<\beta$, $t\geq t_0$ for any solution $x(t)=x(t;t_0,\phi)$ of (1) with $h_0(t_0,\phi)<\alpha$ and the system (1) is $(h_0, h)$ — uniformly bounded. This completes the proof of Theorem 1.

**Corollary 2.** Let the following conditions hold:

1. Conditions 1, 2 and 3 of Theorem 1 are valid.
2. Condition 4 of Theorem 1 is valid for $x\in\Omega_1$.

Then the system (1) is $(h_0, h)$ — uniformly bounded.

**Theorem 2.** Assume the following conditions hold :

1. Assumptions A1—A4 are valid.

2. $h, h^0 \in \Gamma$ and $\bar{h}\,(t,\phi) \le \varphi\,(h_0(t,\phi))$ for some $\varphi \in K$ where $h_0, \bar{h}$ are defined by (3).

3. For $\rho > 0$, there exists $V \in V_0$ such that $a\,(\bar{h}\,(t,\phi)) \le V(t+0,x) \le b\,(h_0(t,\phi))$ for $(t,\ \phi) \in S^c(h_0,\ \rho)$, where $a,\ b \in K$ and $a\,(r) \to \infty$ as $r \to \infty$.

4. For $t \ge t_0, (t, x(t)) \in S^c(h^0, \rho)$ and $x \in \Omega_1$ we have $D\_V(t, x\,(t)) \le 0, t \ne \tau_k$, $V(\tau_k + 0, x\,(\tau_k) + I_k(x\,(\tau_k))) \le V(\tau_k, x\,(\tau_k))$.

Then the system (1) is $(h_0,\ h)$ — uniformly bounded.

The proof of Theorem 2 is analogous to the proof of Theorem 1 and we will omit it.

**Theorem 3.** Assume the following conditions hold :

1. Conditions 1, 2 and 3 of Theorem 2 are valid:

2. For $t \ge t_0$, $(t,\ x_t) \in S^c(h_0,\ \rho)$ and $x \in \Omega_1$ we have $D\_V(t,\ x\,(t)) \le$ $\le -c\,(h_0(t,x_t)),\ t \ne \tau_k, V(\tau_k + 0, x\,(\tau_k) + I_k(x\,(\tau_k))) \le V(\tau_k, x\,(\tau_k))$.

Then the system (1) is $(h_0,\ h)$ — uniformly ultimately bounded.

**Proof.** The system (1) is $(h_0, h)$ — uniformly bounded by means of Theorem 2. Then there exists a positive number $B$ such that for each $t_0 \in R_+$

$$h_0(t_0,\phi) < \delta_0 \text{ implies } h\,(t, x\,(t; t_0,\phi)) < B,\ t \ge t_0.$$

Now we consider the solution $x(t) = x\,(t; t_0, \phi)$ of (1) with $h_0(t_0,\phi) < \alpha$, where $\alpha$ is arbitrary number $\delta_0 > \alpha > \rho_0$. Then there exists a positive number $\beta = \beta(\alpha) >$ $> \max\{\rho_0, \varphi\,(\rho), a^{-1}(b\,(\alpha))\}$ and $\beta < B$ such that $h\,(t, x\,(t)) < \beta,\ t \ge t_0$.

Now let the function $L: R_+ \to R_+$ be a continuous and nondecreasing on $R_+$, and $L(u) > u$ as $u > 0$. We set $\eta = \inf\,\{L(u) - u : a\,(\varphi\,(\rho)) \le u \le a\,(\beta)\}$. Then

$$L(u) > u + \eta \quad \text{as} \quad a\,(\varphi\,(\rho)) \le u \le a\,(\beta) \tag{11}$$

and we choose the integer $\nu$ such that

$$a\,(\varphi\,(\rho)) + \nu\eta > a\,(\beta). \tag{12}$$

If $V(t+0, x\,(t+0)) \ge a\,(\varphi\,(\rho_0))$ for some $t \ge t_0$ then

$$V(t, x\,(t)) \ge V(t+0, x\,(t+0)) \ge a\,(\varphi\,(\rho_0)) \ge a\,(\varphi\,(\rho)),$$
$$b\,(h_0(t,x_t)) \ge V(t+0, x\,(t+0)) \ge a\,(\varphi\,(\rho_0)) \ge a\,(\varphi\,(\rho))$$

and therefore $h_0(t, x_t)) \ge b^{-1}(a\,(\varphi\,(\rho))) = \delta_1$. Hence

$$c\,(h_0(t,x_t)) \ge c(\delta_1) = \delta_2 \tag{13}$$

Let us denote $\xi_k = t_0 + k\,\dfrac{\eta}{\delta_2}$, $k = 0, 1, 2, ..., \nu$. We want to prove

$$V(t, x\,(t)) < a\,(\varphi\,(\rho)) + (\nu - k)\,\eta,\ \ t \ge \xi_k, \tag{14}$$

for all $k = 0,1, 2, ..., \nu$. Indeed, using Corollary 1, condition 3 of Theorem 2 and (12) we obtain

$$V(t,x(t;t_0,\phi)) \le V(t_0+0,\phi(0)) \le$$
$$\le b(h_0(t_0,\phi)) < b(\alpha) < a(\beta) < a(\varphi(\rho)) + \nu\eta, \quad t > t_0 = \xi_0$$

that means the validity of (14) for $k = 0$. Assume (14) to be fulfilled for some integer $k$, $0 < k < \nu$, i. e.

$$V(s,x(s)) < a(\varphi(\rho)) + (\nu - k)\eta, \quad s \ge \xi_k. \tag{15}$$

We suppose now that $V(t,x(t)) \ge a(\varphi(\rho)) + (\nu - k - 1)\eta$, $\xi_k \le t \le \xi_{k+1}$. Then

$$a(\varphi(\rho)) \le V(t,x(t)) \le V(t_0+0,\phi(0)) \le$$
$$\le b(h_0(t_0,\phi)) < b(\alpha) < a(\beta), \; \xi_k \le t \le \xi_{k+1}$$

and (11) and (15) imply

$$L(V(t,x(t))) > V(t,x(t)) + \eta \ge a(\varphi(\rho)) + (\nu - k)\eta > V(s,x(s)), \; \xi_k \le s \le t \le \xi_{k+1}.$$

Therefore $x(\cdot) \in \Omega_1$ as $\xi_k \le s \le t \le \xi_{k+1}$. Then condition 3 of Theorem 3 and (13) yield

$$V(\xi_{k+1}, x(\xi_{k+1})) \le V(\xi_k + 0, x(\xi_k + 0)) - \int_{\xi_k}^{\xi_{k+1}} c(h_0(s,x(s)))\, ds <$$
$$< a(\varphi(\rho)) + (\nu - k)\eta - \delta_2[\xi_{k+1} - \xi_k] = a(\varphi(\rho)) + (\nu - k - 1)\eta < V(\xi_k, x(\xi_k)),$$

which contradicts to the fact that $x(\cdot) \in \Omega_1$ as $\xi_k \le s \le t \le \xi_{k+1}$. Therefore there exists $t^*$, $\xi_k \le t^* \le \xi_{k+1}$ such that $V(t^*, x(t^*)) < a(\varphi(\rho)) + (\nu - k - 1)\eta$ and condition 2 of Theorem 3 implies $V(t^* + 0, x(t^* + 0)) < a(\varphi(\rho)) + (\nu - k - 1)\eta$. We will prove $V(t, x(t)) < a(\varphi(\rho)) + (\nu - k - 1)\eta$, $t \ge t^*$. Supposing the opposite, we set

$$\mu = \inf\{t \ge t^* : V(t, x(t)) \ge a(\varphi(\rho)) + (\nu - k - 1)\eta\}.$$

It follows from the condition 2 of Theorem 3 that $\mu \ne \tau_k$, $k = 1, 2, ...$, whence $V(\mu, x(\mu)) = a(\varphi(\rho)) + (\nu - k - 1)\eta$. Then for sufficiently close to zero $\sigma < 0$ we have

$$V(\mu + \sigma, x(\mu + \sigma)) < a(\varphi(\rho)) + (\nu - k - 1)\eta,$$

whence $D\_V(\mu, x(\mu)) \ge 0$. On the other hand, we can prove as above that $x(\cdot) \in \Omega_1$ as $t^* \le s \le t \le \mu$ and therefore $D\_V(\mu, x(\mu)) \le -\delta_2 < 0$.

The contradiction we have already obtained yields

$$V(t,x(t)) < a(\varphi(\rho)) + (\nu - k - 1)\eta, \quad t \ge \xi_{k+1}.$$

It follows that (14) holds for all $k = 0, 1, 2, ..., \nu$. Let $T = T(\alpha) = \nu \frac{\eta}{\delta_2}$. Then (14) implies

$$V(t, x(t)) < a(\varphi(\rho)) \text{ as } t \geq t_0 + T. \tag{16}$$

Finally, condition 3 of Theorem 2 and (16) lead us to

$$a(h(t, x(t))) \leq a(\bar{h}(t, x_t)) \leq V(t+0, x(t+0)) \leq$$

$$\leq V(t, x(t)) < a(\varphi(\rho)) < a(\beta) < a(B) \text{ as } t \geq t_0 + T.$$

Therefore $h_0(t_0, \phi) < \alpha$ implies $h(t, x(t)) < B$ as $t \geq t_0 + T$ and (1) is a $(h_0, h)$ — uniformly ultimately bounded system. The proof is completed.

Досліджено задачу з початковими умовами для імпульсних функціональних диференціальних рівнянь з кінцевим запізнюванням та фіксованими моментами імпульсного впливу. У результаті використання кусково-неперервних функцій та методу Разуміхіна знайдено достатні умови обмеженості відносно двох різних кусково-неперервних критеріїв для таких рівнянь. Наведені результати доповнюють та підтверджують такі, що отримано раніше.

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