J. Duan
Department of Applied Mathematics
Illinois Institute of Technology
(Chicago, IL 60616, USA, E-mail: duan@iit.edu)

Predictability in Spatially Extended Systems with Model Uncertainty *. I

Macroscopic models for spatially extended systems under random influences are often described by stochastic partial differential equations. Some techniques for understanding solutions of such equations, such as estimating correlations, Liapunov exponents and impact of noises, are discussed. They are relevant for understanding predictability in spatially extended systems with model uncertainty, for example, in physics, geophysics and biological sciences. The presentation is for a wide audience.

Рассмотрены некоторые методы представления решений стохастических дифференциальных уравнений в частных производных, в частности в задачах корреляции оценки, экспоненты Ляпунова и воздействие шумов. Методы пригодны для понимания предсказуемости в пространственно распределенных системах с неопределенностью модели, например, в физике, геофизике и биологических науках.

Key words: stochastic partial differential equations, correlation, Liapunov exponents, predictability, uncertainty, invariant manifolds, impact of noise

1. Motivation. Scientific and engineering systems are often subject to uncertainty or random influence. Randomness can have delicate impact on the overall evolution of such systems, for example, stochastic bifurcation [1], stochastic resonance [2], and noise-induced pattern formation [3]. Taking stochastic effects into account is of central importance for the development of mathematical models of complex phenomena in engineering and science.

Macroscopic models for systems with spatial dependence («spatially extended») are often in the form of partial differential equations (PDEs). Randomness appears in these models as stochastic forcing, uncertain parameters, random sources or inputs, and random boundary conditions (BCs). These models are usually called stochastic partial differential equations (SPDEs). Note that SPDEs may also serve as intermediate «mesoscopic» models in some multiscale systems. Although we may think that SPDEs could be reduced to large systems

* This work was partly supported by the NSF Grants 0542450 and 0620539.

of stochastic ordinary differential equations (SODEs) in numerical approaches [4, 5], it is beneficial to work on SPDEs directly when dealing with some dynamical issues [6—18].

There is a growing recognition of a role for the inclusion of stochastic terms in the modeling of complex systems. For example, there has been increasing interest in mathematical modeling via SPDEs, for the climate system, condensed matter physics, materials sciences, mechanical and electrical engineering, and finance, to name just a few. The inclusion of stochastic effects has led to interesting new mathematical problems at the interface of dynamical systems, partial differential equations, scientific computing, and probability theory. Problems arising in the context of stochastic dynamical modeling have inspired interesting research topics about, for example, the interaction between noise, nonlinearity and multiple scales, and about efficient numerical methods for simulating random phenomena.

There has been some promising new development in understanding dynamics of SPDEs via invariant manifolds [19—22] and stochastic homogenization [23, 24]. But we will not discuss these issues in this paper. For general background on SPDEs, see [25—29].

Although some progress has been made in SPDEs in the past decade, many challenges remain and new problems arise in modeling basic mechanisms in complex systems under uncertainty. These challenging problems include overall impact of noise, stochastic bifurcation, ergodic theory, invariant manifolds, and predictability of dynamical behavior, to name just a few. Solutions for these problems will greatly enhance our ability in understanding, quantifying, and managing uncertainty and predictability in engineering and science. Breakthroughs in solving these challenging problems are expected to emerge.

This paper is organized as follows. After reviewing some basic concepts on probability in Hilbert space in part 2, we discuss stochastic analysis and SPDEs in part 3. Then we derive correlations of some linear SPDEs, Lyapunov exponents, and the impact of uncertainty in parts 4, 5 and 6, respectively.

2. Stochastic Tools in Hilbert Space. *Hilbert space*. Recall that the Euclidean space $\mathbb{R}^n$ is equipped with the usual metric or distance

$$d(x, y) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2},$$

norm or length

$$\| x \| = \sqrt{\sum_{j=1}^{n} x_j^2},$$

and the usual scalar product

$$x \cdot y = < x, y > = \sum_{j=1}^{n} x_j \, y_j \; .$$

The Borel σ-field of $\mathbb{R}^n$, i. e., $\mathcal{B}(\mathbb{R}^n)$ is generated by all open balls in $\mathbb{R}^n$.

Hilbert space H is a set with three mathematical operations: scalar multiplication, addition and scalar product $\langle \cdot, \cdot \rangle$, satisfying the usual properties as we are familiar with in elementary mathematics. The scalar product induces a natural norm $\| u \| = \sqrt{\langle u, u \rangle}$. The Borel σ-field of H, i. e., $\mathcal{B}(H)$ is generated by all open balls in H.

Probability in Hilbert space. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with sample space Ω, σ-field $\mathcal{F}$ and probability measure $\mathbb{P}$. Consider a random variable in Hilbert space H (i.e., taking values in H):

$$X : \Omega \to H.$$

Its mean or mathematical expectation is defined in terms of the integral with respect to the probability measure $\mathbb{P}$:

$$\mathbb{E}(X) = \int_{\Omega} X(\omega) \, d\mathbb{P}(\omega).$$

Its variance is:

$$Var(X) = \mathbb{E}\langle X - \mathbb{E}(X), X - \mathbb{E}(X) \rangle = \mathbb{E}\| X - \mathbb{E}(X) \|^2 = \mathbb{E}\| X \|^2 - \| \mathbb{E}(X) \|^2 .$$

Especially, if $\mathbb{E}(X) = 0$, then $Var(X) = \mathbb{E}\|X\|^2$. Covariance operator of X is defined as

$$Cov(X) = \mathbb{E}[(X - \mathbb{E}(X)) \otimes (X - \mathbb{E}(X))],$$

where for any a, $b \in H$, we denote $a \otimes b$ the linear operator in H defined by

$$a \otimes b : H \to H, \;\; (a \otimes b)\, h = a \langle b, h \rangle , \;\; h \in H.$$

Let X and Y be two random variables taking values in Hilbert space H. The correlation operator of X and Y is defined by

$$Cor(X, Y) = \mathbb{E}[(X - \mathbb{E}(X)) \otimes (Y - \mathbb{E}(Y))].$$

Remark 1. $Cov(X)$ is a symmetric positive and trace-class linear operator with trace

$$Tr\, Cov(X) = \mathbb{E}\langle X - \mathbb{E}(X), X - \mathbb{E}(X) \rangle = \mathbb{E}\| X - \mathbb{E}(X) \|^2 .$$

Moreover, $Tr\, Cor(X, Y) = \mathbb{E}\langle X - \mathbb{E}(X), Y - \mathbb{E}(Y) \rangle$.

Gaussian random variables. Recall that a random variable taking values in $\mathbb{R}^n$

$$X : \Omega \to \mathbb{R}^n$$

is called Gaussian, if for any $a = (a_1, ..., a_n) \in \mathbb{R}^n$, $X \cdot a = a_1 X_1 + ... + a_n X_n$ is a scalar Gaussian random variable. A Gaussian random variable in $\mathbb{R}^n$ is denoted as $X \sim \mathbb{N}(m, Q)$, with mean vector m and covariance matrix Q. The covariance matrix Q is symmetric and non-negative (i. e., eigenvalue $\lambda_j \geq 0$, $j = 1, ..., n$). The trace of Q is written as $Tr(Q) = \lambda_1 + ... + \lambda_n$. The covariance matrix is defined as

$$Q = (Q_{ij}) = (\mathbb{E}[(X_i - m_i)(X_j - m_j)]) .$$

We use the notations $E(X) = m$ and $Cov(X) = Q$. The probability density function for this Gaussian random variable X in $\mathbb{R}^n$ is

$$f(x) = f(x_1, ..., x_n) = \frac{\sqrt{\det(A)}}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum_{j,k=1}^{n}(x_j - m_j) a_{jk} (x_k - m_k)},$$

where $A = Q^{-1} = (a_{jk})$.

The probability distribution function of X is

$$F(x) = \mathbb{P}(\omega : X(\omega) \leq x) = \int_{-\infty}^{x} f(x)\, dx.$$

The probability distribution measure μ (or law $\mathcal{L}_X$) of X is:

$$\mu(B) = \int_B f(x)\, dx, \; B \in \mathcal{B}(\mathbb{R}^n).$$

Here are some observations. For $a, b \in \mathbb{R}^n$,

$$\mathbb{E}\langle X, a\rangle = \mathbb{E}\sum_{i=1}^{n} a_i X_i = \sum_{i=1}^{n} a_i \mathbb{E}(X_i) = \sum_{i=1}^{n} a_i m_i = \langle m, a\rangle$$

$$\mathbb{E}(\langle X - m, a\rangle\langle X - m, b\rangle) = \mathbb{E}\left(\sum_i a_i (X_i - m_i) \sum_j b_j (X_j - m_j)\right) =$$

$$= \sum_{i,j} a_i b_j \mathbb{E}[(X_i - m_i)(X_j - m_j)] = \sum_{i,j} a_i b_j Q_{ij} = \langle Qa, b\rangle .$$

In particular, $\langle Qa, a\rangle = \mathbb{E}\langle X - m, a\rangle^2 \geq 0$, which confirms that Q is nonnegative. Also, $\langle Qa, b\rangle = \langle a, Qb\rangle$, which implies that Q is symmetric.

Definition 1. A random variable $X: \Omega \to H$ in Hilbert space H is called a Gaussian random variable and denoted as $X \sim \mathbb{N}(m, Q)$, if for every a in H, the real random variable $\langle X, a \rangle$ is a scalar Gaussian random variable (i. e., taking values in $\mathbb{R}^1$).

Remark 2. If X is a Gaussian random variable taking values in Hilbert space H, then for all a, $b \in H$,

(i) Mean vector $\mathbb{E}(X) = m : \mathbb{E}\langle X, a \rangle = \langle m, a \rangle$;

(ii) Covariance operator $Cov\,(X) = Q : \mathbb{E}(\langle X - m, a \rangle \langle X - m, b \rangle) = \langle Qa, b \rangle$

Remark 3. The Borel probability measure μ on $(H, \mathcal{B}(H))$ induced by a Gaussian random variable X taking values in Hilbert space H, is called a Gaussian measure. If μ is a Gaussian measure in H, then there exist an element $m \in H$ and a non-negative symmetric continuous linear operator $X : \Omega \to H$ such that: For all h, h_1, $h_2 \in H$,

(i) Mean vector m: $\int\limits_H \langle h, x \rangle \, d\mu(x) = \langle m, h \rangle$;

(ii) Covariance operator Q: $\int\limits_H \langle h_1, x \rangle \langle h_2, x \rangle \, d\mu(x) - \langle m, h_1 \rangle \langle m, h_2 \rangle = \langle Qh_1, h_2 \rangle$.

Since the covariance operator Q is non-negative and symmetric, the eigenvalues of Q are non-negative and the eigenvectors e_n's form an orthonormal basis for Hilbert space H: $Qe_n = q_n e_n$, $n = 1, 2, \ldots$. Moreover, trace $Tr\,(Q) = \sum\limits_{n=1}^{\infty} q_n$.

Note that

$$X - m = \sum X_n e_n$$

with coefficients $X_n = \langle X - m, e_n \rangle$,

$$\mathbb{E}\, X_n^2 = \mathbb{E}(\langle X - m, e_n \rangle \langle X - m, e_n \rangle) = \langle Qe_n, e_n \rangle = \langle q_n e_n, e_n \rangle = q_n .$$

Therefore,

$$\|X - m\|^2 = \sum X_n^2,$$

$$\mathbb{E}\|X - m\|^2 = \sum \mathbb{E} X_n^2 = \sum q_n = Tr\,(Q) .$$

We use $L^2(\Omega, H)$, or just $L^2(\Omega)$, to denote the (new) Hilbert space of square-integrable random variables $x : \Omega \to H$. In Hilbert space $L^2(\Omega, H)$, the scalar product is $< x, y > = \mathbb{E} < x(\omega), y(\omega) >$, where $\mathbb{E}$ denotes the mathematical expectation (or mean) with respect to probability $\mathbb{P}$. This scalar product induces the usual mean square norm $\| x \| := \sqrt{\mathbb{E}\|x(\omega)\|^2}$, which provides an appropriate convergence concept.

Brownian motion. Recall that a Brownian motion (or Wiener process) $W(t)$, also denoted as W_t, in R^n is a Gaussian stochastic process on a underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a sample space, $\mathcal{F}$ is a σ-field composed of measurable subsets of Ω (called «events»), and $\mathbb{P}$ is a probability (also called probability measure). Being a special Gaussian process, W_t is characterized by its mean vector (taking to be the zero vector) and its covariance operator, a $n \times n$ symmetric positive definite matrix (taking to be the identity matrix). More specifically, W_t satisfies the following conditions [30]:

(a) $W(0) = 0$ a.s.,

(b) W has continuous paths or trajectories a.s.,

(c) W has independent increments,

(d) $W(t) - W(s) \sim N(0, (t-s)I)$, t and $s > 0$ and $t \geq s \geq 0$, where I is the $n \times n$ identity matrix. The Brownian motion in R^1 is called a scalar Brownian motion.

Remark 4. (i) The covariance operator here is a constant $n \times n$ identity matrix I, i. e., $Q = I$ and $Tr\,(Q) = n$.

(i) $W(t) \sim N(0, tI)$, i.e. $W(t)$ has probability density function

$$p_t(x) = \frac{1}{(2\pi t)^{n/2}}\, e^{-\frac{x_1^2 + \ldots + x_n^2}{2t}}.$$

(ii) For every $\alpha \in \left(0, \frac{1}{2}\right)$, for a.e. $\omega \in \Omega$ there exists $C(\omega)$ such that

$$|W(t,\omega) - W(s,\omega)| \leq C(\omega)\, |t-s|^{\alpha},$$

namely, Brownian paths are Hölder continuous with exponent less than one half.

Note that the generalized time derivative of Brownian motion W_t is a mathematical model for white noise [31].

Now we define Wiener process, or Brownian motion, in Hilbert space U. We consider a symmetric nonnegative linear operator Q in U. If the trace $Tr\,(Q) < +\infty$, we say Q is a trace class (or nuclear) operator. Then there exist a complete orthonormal system (eigenfunctions) $\{e_k\}$ in U, and a (bounded) sequence of nonnegative real numbers (eigenvalues) q_k such that $Qe_k = q_k e_k$, $k = 1, 2, \ldots$. A stochastic process $W(t)$, or W_t, taking values in U for ≥ 0, is called *a* Wiener process with covariance operator Q if :

(a) $W(0) = 0$ a.s.,

(b) W has continuous trajectories a.s.,

(c) W has independent increments,

(d) $W(t) - W(s) \sim N(0, (t-s)\,Q)$, $t \geq s$:

Hence, $\mathbb{E}W(0) = 0$ and $Cov\,(W(t)) = tQ$.

We can think the covariance matrix Q as a $\infty\times\infty$ diagonal matrix, with diagonal elements $q_1, q_2, \ldots, q_n, \ldots$. For any $a \in H$,

$$a=\sum_n < a, e_n > e_n,$$

$$Qa=\sum_n < a, e_n > Qe_n = \sum_n q_n < a, e_n > e_n .$$

We define, for $\gamma > 0$, especially for $\gamma \in (0; 1)$,

$$Q^\gamma a=\sum_n q_n^\gamma < a, e_n > e_n,$$

when the right hand side is defined.

Representains of Brownian motion in Hilbert space. It is known that W_t has an infinite series representation [25]:

$$W_t(\omega)=\sum_{n=1}^{\infty}\sqrt{q_n}W_n(t)e_n,$$

where

$$W_n(t):=\begin{cases}\dfrac{\langle W(t), e_n\rangle}{\sqrt{q_n}}, q_n>0,\\ 0, \qquad\qquad q_n=0.\end{cases}$$

are the standard scalar independent Brownian motions. Namely, $W_n(t)\sim \mathbb{N}(0,t)$, $\mathbb{E}W_n(t)=0$, $\mathbb{E}W_n(t)^2=t$ and $\mathbb{E}W_n(t)W_n(s)=\min(t,s)$. This infinite series converges in $L^2(\Omega)$, as long as $Tr(Q)=\sum q_n<\infty$.

Remark 5. For example in $H=L^2(0,1)$, we have an orthonormal basis $e_n=\sin(n\pi x)$. In the above infinite series representation, taking derivative with respect to x, we get

$$\partial_x W_t(\omega)=\sum_{n=1}^{\infty}\sqrt{2}(n\pi)\sqrt{q_n}W_n(t)\cos(n\pi x).$$

In order for this series to converge, we need $\sqrt{2}(n\pi)\sqrt{q_n}$ converges to zero sufficiently fast as $n\to\infty$. So q_n being small helps. In this sense, the trace $Tr(Q)=\sum q_n$ may be seen as a measurement for spatial regularity of white noise $\dot{W}_t$ the smaller the trace $Tr(Q)$, the more regular of the noise.

We do some calculations. For $a, b \in H$, we have the following identities:

$$\mathbb{E}\langle W_t, W_t\rangle=\mathbb{E}\|W_t\|^2=\mathbb{E}\left\langle\sum_{n=1}^{\infty}\sqrt{q_n}W_n(t)\,e_n,\ \sum_{n=1}^{\infty}\sqrt{q_n}W_n(t)e_n\right\rangle=$$

$$=\sum_{n=1}^{\infty} q_n \mathbb{E}\langle W_n(t), W_n(t)\rangle = t\sum_{n=1}^{\infty} q_n = tTr(Q),$$

$$\mathbb{E}\langle W_t, a\rangle = \langle 0, a\rangle = 0,$$

$$\mathbb{E}(\langle W_t, a\rangle\langle W_t, b\rangle) = \mathbb{E}\left[\left\langle \sum_{n=1}^{\infty}\sqrt{q_n} W_n(t)\, e_n, a\right\rangle \left\langle \sum_{n=1}^{\infty}\sqrt{q_n} W_n(t)\, e_n, b\right\rangle\right] =$$

$$= \mathbb{E}\sum_{m,n}\sqrt{q_m q_n} W_m(t) W_n(t) < e_m, a >< e_n, b > =$$

$$= \sum_n t q_n < e_n, a >< e_n, b > = t\sum_n < e_n, a >< q_n e_n, b > =$$

$$= t\sum_n < e_n, a >< Q e_n, b > = t\sum_n < Q < e_n, a > e_n, b > =$$

$$= t < Q\sum_n < e_n, a > e_n, b > = t < Qa, b >,$$

where we have used the fact that $a = \sum_n < e_n, a > e_n$ in the final step. In particular, taking $a = b$, we obtain

$$\mathbb{E}\langle W_t, a\rangle^2 = t\langle Qa, a\rangle, \ Var(\langle W_t, a\rangle) = t\langle Qa, a\rangle.$$

More generally, $\mathbb{E}(\langle W_t, a\rangle\langle W_s, b\rangle) = \min(t, s) < Qa, b >$. Moreover,

$$\mathbb{E}[W_t(x) W_s(y)] = \mathbb{E}\left\{\sum_{n=1}^{\infty}\sqrt{q_n} W_n(t)\, e_n(x)\sum_{m=1}^{\infty}\sqrt{q_m} W_m(s)\, e_m(y)\right\} =$$

$$= \sum_{n,m=1}^{\infty}\sqrt{q_n q_m}\mathbb{E}[W_n(t) W_m(s)]\, e_n(x)\, e_m(y) =$$

$$= \min(t, s)\sum_{n=1}^{\infty} q_n e_n(x)\, e_n(y) = \min(t, s)\, q(x, y),$$

where

$$q(x, y) = \sum_{n=1}^{\infty} q_n e_n(x)\, e_n(y).$$

On the other hand, the covariance operator may be represented in terms of $q(x, y)$:

$$Qa = Q\sum_n < e_n, a > e_n = \sum_n < e_n, a > Q e_n = \sum_n < e_n, a > q_n e_n =$$

$$= \sum_n \int_0^1 a(y)\, e_n(y)\, dy q_n e_n(x) = \int_0^1 q(x, y)\, a(y)\, dy.$$

Sometimes we call the kernel function $q(x,y)$ the spatial correlation. The smoothness of $q(x,y)$ depends on the decaying property of q_n 's.

3. Stochastic Partial Differential Equations. *Stochastic calculus in Hilbert space.* We define the Ito stochastic integral:

$$\int_0^T \Phi(s,\omega)\, dW_s.$$

Note that since W_t takes values in Hilbert space U. The integrand $\Phi(t,\omega)$ is usually a linear operator from U to H (for each time t and each sample ω):

$$\Phi : U \to H.$$

It is also possible to take W_t as a scalar, real-valued Brownian motion. For example, in $\int_0^T u(s)\, dW_s$ if W_t is a scalar Brownian motion, we can interpret the integrand u as a multiplication operator. For Brownian motion W_t in U

$$W_t(\omega) = \sum_{n=1}^{\infty} \sqrt{q_n} W_n(t)\, e_n,$$

we define

$$\int_0^T \Phi(s,\omega)\, dW_s(\omega) = \sum_{n=1}^{\infty} \sqrt{q_n} \int_0^T \Phi(s,\omega)\, e_n dW_n(s).$$

A property of Ito integrals:

$$\mathbb{E} \int_0^T \Phi(s,\omega)\, dW_s(\omega) = 0\,.$$

Deterministic calculus in Hilbert space. In order to discuss more tools to handle stochastic calculus in Hilbert space, we need to recall some concepts of deterministic calculus. For calculus in Euclidean space $\mathbb{R}^n$, we have concepts derivative and directional derivative. In Hilbert space, we have the corresponding Fréchet derivative and Gateaux derivative [32, 33].

Let H and $\hat{H}$ be two Hilbert spaces, and $F : U \subset H \to \hat{H}$ be a map, whose domain of definition U is an open subset of H. Let $L(H, \hat{H})$ be the set of all bounded linear operators $A : H \to \hat{H}$. In particular, $L(H) := L(H,H)$. We can also introduce a multilinear operator $A_1: H \times H \to \hat{H}$. The space of all these multilinear operators is denoted as $L(H \times H, \hat{H})$.

Definition 2. The map F is Fréchet difierentiable at $u_0 \in U$ if there is a linear bounded operator $A: H \to \hat{H}$ such that

$$\lim_{h \to 0} \frac{\| F(u_0 + h) - F(u_0) - Ah \|}{\| h \|} = 0,$$

i. e. $\| F(u_0 + h) - F(u_0) - Ah \| = o(\| h \|)$, where $\| \cdot \|$ denotes norms in H or $\hat{H}$ as appropriate. The linear bounded operator A is called the Fréchet derivative of F at u_0, and is denoted as $F_u(u_0)$, or sometimes $F'(u_0)$.

If F is linear, its Fréchet derivative is itself.

Definition 3. The directional derivative of F at $u_0 \in U$ in the direction $h \in H$ is defined by the limit

$$\delta F(u_0; h) := \lim_{t \to 0} \frac{F(u_0 + th) - F(u_0)}{t}.$$

If this limit exists for every $h \in H$, and $F'_G(u_0)\, h := \delta F(u_0; h)$ is a linear map, then we say that F is Gateaux differentiable at u_0. The linear map $F'_G(u_0)$ is called the Gateaux derivative of F at u_0.

In fact, if F is Fréchet differentiable at u_0, then it is also Gateaux differentiable at u_0 and they are equal [32, 33]: $F_u(u_0) = F'_G(u_0)$. But the converse is not usually true. It is true under suitable conditions [32, p. 68].

For any nonlinear map $F: U \subset H \to Y$, its Fréchet derivative $F'(u_0)$ is a linear operator, i.e. $F'(u_0) \in L(H, Y)$. Similarly, we can define higher order Fréchet derivatives. Each of these derivatives is a multilinear operator. For example,

$$f''(u_0): H \times H \to Y,$$
$$(h, k) \rightarrowtail f''(u_0)(h, k).$$

We denote

$$f''(u_0)\, h^2 := f''(u_0)(h, h),$$
$$f'''(u_0)\, h^3 := f'''(u_0)(h, h, h),$$

and similarly for higher order derivatives. Then we have the Taylor expansion in Hilbert space

$$f(u + h) = f(u) + f'(u)\, h + \frac{1}{2!} f''(u)\, h^2 + \ldots + \frac{1}{m!} f^{(m)}(u)\, h^m + R_{m+1}(u, h),$$

where the remainder

$$R_{m+1}(u, h) = \frac{1}{(m+1)!} \int_0^1 (1 - s)^m f^{(m+1)}(u + sh)\, h^{m+1}\, ds.$$

Remark 6. It is interesting to relate these two concepts with the classical concept of variational derivative (or functional derivative) that is used in the context of calculus of variations. The variational derivative is usually considered for functionals defined as spatial integrals, such as a Langrange functional in mechanics. For example,

$$F(u)=\int_0^l G(u(x),\, u_x(x))\, dx,$$

where u is defined on $x\in[0,l]$ and satisfies zero Dirichlet boundary condition at $x = 0$, l. Then it is known [34] that

$$F_u(u)\, h=\int_0^l \frac{\delta F}{\delta u} h(x)\, dx, \tag{1}$$

for h in the Hilbert space $H_0^1(0,l)$. The quantity $\dfrac{\delta F}{\delta u}$ is the classical variational derivative of F. The equation (31) above gives the relation between Fréchet derivative and variational derivative.

Ito's formula in Hilbert space. We get back to stochastic calculus in Hilbert space H. We first look at the Ito's formula; see [25] or [29].

Theorem 1. Let u be the solution of the SPDE

$$du=b(u)\,dt+\Phi(u)\,dW_t,\;\; u(0)=u_0.$$

Assume that $F(t, u)$ be a given smooth (deterministic) function:

$$F:\,[0,\infty)\times H\to\mathbb{R}^1.$$

Then

(i) Ito's Formula: Difierential form

$$dF(t,u(t))=F_u(t,u(t))(\Phi(u(t))\,dW_t)+\{F_t\,u(t))+F_u(t,u(t))(b(u(t)))+$$
$$+\frac{1}{2}Tr\,[F_{uu}(t,u(t))(\Phi(u(t))\,Q^{\frac{1}{2}})(\Phi(u(t))\,Q^{\frac{1}{2}})^*\,]\}\,dt\,,$$

where F_u and F_{uu} are Fréchet derivatives, F_t is the usual partial derivative in time, and *denotes adjoined operator. This formula is understood with the following symbolic operations in mind:

$$\langle dt,\, dW_t\rangle=\langle dt,\, dW_t\rangle=0, \langle dW_t,\, dW_t\rangle=Tr\,(Q)\,dt.$$

(ii) Ito's Formula: Integral form

$$F(t,\,u(t))=F(0,\,u(0))+\int_0^t F_u(s,u(s))(\Phi(u(s))\,dW_s)+$$

$$+\int_0^t \Big\{ F_t(s,u(s)) + F_u(s,u(s))(b(u(s))) +$$

$$+\frac{1}{2} Tr\left[F_{uu}(s,u(s))(\Phi(u(s))Q^{\frac{1}{2}})(\Phi(u(s))Q^{\frac{1}{2}})^* \right]\Big\} ds,$$

where F_u and F_{uu} are Fréchet derivatives and F_t is the usual partial derivative in time. Moreover,

$$\int_0^t F_u(s,u(s))(\Phi(u(s))\,dW_s) = \int_0^t \widetilde{\Phi}(u(s))\,dW_s$$

and for all s, $v \in H$, $\omega \in \Omega$ the operator $\widetilde{\Phi}(u(s))$ is defined by

$$\widetilde{\Phi}(u(s))(v) := F_u(s,u(s))(\Phi(u(s))\,v).$$

Also,

$$Tr\left[F_{uu}(s,u(s))(\Phi(u(s))Q^{\frac{1}{2}})(\Phi(u(s))Q^{\frac{1}{2}})^* \right] =$$

$$= Tr\left[(\Phi(u(s))Q^{\frac{1}{2}})^* F_{uu}(s,u(s))\ (\Phi(u(s))Q^{\frac{1}{2}}) \right].$$

Note that for the symmetric non-negative covariance operator Q with eigenvalues $q_n \geq 0$ and eigenvector e_n, $n = 1, 2, \ldots$, we have

$$Qu = \sum_n q_n \langle u, e_n \rangle\, e_n, \quad Q^{\frac{1}{2}} u = \sum_n q_n^{\frac{1}{2}} \langle u, e_n \rangle\, e_n.$$

In fact, for a given function $h : \mathbb{R} \to \mathbb{R}$, we define the operator $h(Q)$ through the following natural formula [35, p. 293—294],

$$h(A)\,u = \sum_n h(q_n) \langle u, e_n \rangle\, e_n,$$

when the right hand side is defined.

Example 1. A typical application of Ito's formula for SPDEs

$$du = b(u)\,dt + \Phi(u)\,dW_t, \ \ u(0) = u_0.$$

Take Hilbert space $H = L^2(D)$, $D \subset \mathbb{R}^n$ with the usual scalar product $\langle u,v \rangle = \int_D uv dx$. Energy functional

$$F(u) = \frac{1}{2}\int_D u^2 dx = \frac{1}{2}\|u\|^2.$$

In this case,

$$F_u(u)(h) = \int_D uhdx,$$

and

$$F_{uu}(u)(h,k) = \int_D h(x)\,k(x)\,dx,$$

$$\frac{1}{2} d\,\|u\|^2 = \left\{ \langle u, b(u) \rangle + \frac{1}{2} Tr \int_D [(\Phi(u) Q^{\frac{1}{2}})(\Phi(u) Q^{\frac{1}{2}})^*]\,dx \right\} dt + \langle u, \Phi(u)\,dW_t \rangle .$$

Integrating and taking mathematical expectation, we obtain

$$\frac{1}{2}\mathbb{E}\|u\|^2 = \frac{1}{2}\mathbb{E}\|u(0)\|^2 + \mathbb{E}\int_0^t \langle u, b(u)\rangle\,dt +$$

$$+ \frac{1}{2}\mathbb{E}\int_0^t Tr\int_D \left[(\Phi(u(r)) Q^{\frac{1}{2}})(\Phi(u(r)) Q^{\frac{1}{2}})^* \right] dxdr.$$

Note that in this special case, F_u is a bounded operator in $L(H, \mathbb{R})$, which can be identified with H itself due to the Riesz representation theorem.

Example 2. Energy functional

$$F(u) = \int_D |u|^{2p}\,dx = \int_D (|u|^2)^p\,dx$$

for $p \in [1, \infty)$. In this case,

$$F_u(u_0)(h) = 2p\int_D |u_0|^{2p-2} u_0 h dx$$

and

$$F_{uu}(u_0)(h,k) = 2p\int_D |u_0|^{2p-2} hkdx + 4p(p-1)\int_D |u_0|^{2p-4} u_0 h(x)\,u_0 k(x)\,dx.$$

Example 3 [28, p. 153]. Let H be a Hilbert space with scalar product $< \cdot,\cdot >$ and norm $\|\cdot\|^2 = < \cdot,\cdot >$. Consider an energy functional $F(u) = \|u\|^{2p}$ for $p \in [1, \infty)$. In this case,

$$F_u(u_0)(h) = 2p\|u_0\|^{2p-2} < u_0, h >$$

and

$$F_{uu}(u_0)(h,k) = 2p\,\|u_0\|^{2p-2} < h,k > + 4p(p-1)\|u_0\|^{2p-4} < u_0, h >< u_0, k > =$$

$$= 2p\,\|u_0\|^{2p-2} < h,k > + 4p(p-1)\|u_0\|^{2p-4} < (u_0 \otimes u_0)\,h, k >,$$

where $(a \otimes b)\,h := a < b, h >$ (see part 2 or [25, p. 25]).

S t o c h a s t i c p r o d u c t r u l e. Let u and v be solutions of two SPDEs. Then

$$d(uv) = udv + (du)v + dudv.$$

I t o i s o m e t r y:

$$\mathbb{E}\left\| \int_0^t \Phi(t,\omega)\, dW_t \right\|^2 = \mathbb{E}\int_0^t Tr\left[\left(\Phi(r) Q^{\frac{1}{2}} \right) \left(\Phi(r) Q^{\frac{1}{2}} \right)^* \right] dr.$$

G e n e r a l i z e d I t o i s o m e t r y:

$$\mathbb{E}\left\langle \int_0^a F(t,\omega) dW_t, \int_0^b G(t,\omega)\, dW_t \right\rangle_t = \mathbb{E}\int_0^{a\wedge b} Tr\left[\left(G(r,\omega) Q^{\frac{1}{2}} \right) \left(F(r,\omega) Q^{\frac{1}{2}} \right)^* \right] dr,$$

where $a \wedge b = \min(a, b)$.

Stochastic partial differential equations. A general class of SPDEs may be written as $du_t = [Au + f(u)]\, dt + G(u)\, dW_t$, where Au is the linear part, $f(u)$ is the nonlinear part, $G(u)$ the noise intensity (usually an operator), and W_t a Brownian motion. When G depends on u, $G(u)\, dW_t$ is called a multiplicative noise, otherwise it is an additive noise.

For general background on SPDEs, such as wellposedness and basic properties of solutions, see [25, 26, 29].

(*The completion of the paper see in the next issue*)

Розглянуто деякі методи представлення розв'язків стохастичних диференціальних рівнянь у частинних похідних, зокрема у задачах кореляції оцінки, експоненти Ляпунова та впливу шумів. Методи придатні для розуміння передбачуваності у просторово розподілених системах з невизначеністю моделі, наприклад, у фізиці, геофізиці та біологічних науках.

1. *Caraballo T., Langa J., Robinson J. C.* A Stochastic Pitchfork Bifurcation in a Reaction-difiusion Equation// Proc. R. Soc. A. — 2001. — **457.** — P. 2041—2061.
2. *Imkeller P., Pavlyukevich I.* Model Reduction and Stochastic Resonance. Stochastics and Dynamics. — 2002. — **2**, — N 4.
3. *Garcia-Ojalvo J., Sancho J. M.* Noise in Spatially Extended Systems. — Springer-Verlag, 1999.
4. *Millet A., Morien P.-L.* On Implicit and Explicit Discretization Schemes for Parabolic SPDEs in any Dimension// Stochastic Process. Appl. — 2005. — **115**, N 7. — P. 1073—1106.
5. *Allouba H.* SDDEs Limits Solutions to Sublinear Reaction-difiusion SPDEs//Electron. J. Diff. Eqns. — 2003. — N 111. — 21 p.
6. *Bongolan-Walsh V. P., Duan J., Ozgokmen T.* Dynamics of Transport under Random Fluxes on the Boundary// Communications in Nonlinear Science and Numerical Simulation. — 2008. — **13.** — P. 1627—1641.

7. *Cessi P., Louazel S.* Decadal Oceanic Response to Stochastic Wind Forcing// J. Phys. Oceanography. — 2001. — **31.** — P. 3020—3029.
8. *DelSole T.* Stochastic Models of Quasigeostrophic Turbulence// Surveys in Geophysics. — 2004. — **25.** — P. 107—149.
9. *Duan J., Gao H., Schmalfuss B.* Stochastic Dynamics of a Coupled Atmosphere-ocean Model// Stochastics and Dynamics. — 2002. — **2.** — P. 357—380.
10. *Farrell B. F., Ioannou P. J.* Perturbation Growth and Structure in Uncertain Ows. I, II// J. Atmospheric Sci. — 2002. — **59.** — N 18. — P. 2629—2646, 2647—2664.
11. *Grifia A., Castellari S.* Nonlinear General Circulation of an Ocean Model Driven by Wind with a Stochastic Component// J. Marine Research. — 1991. — **49.** — P. 53—73.
12. *Holloway G.* Ocean Circulation: Flow in Probability under Statistical Dynamical Forcing, in Stochastic Models in Geosystems. Ed. by S. Molchanov and W. Woyczynski. — New York: Springer-Verlag, 1996.
13. *Imkeller P., Monahan A. (eds.)* Conceptual Stochastic Climate Models.Special Issue: Stochastics and Dynamics, 2002. — **2.** — N. 3.
14. *Muller P.* Stochastic Forcing of Quasi-geostrophic Eddies// Stochastic Modelling in Physical Oceanography. Ed. by R. J. Adler, P. Muller and B. Rozovskii. — Birkhauser, 1996.
15. *North G. R., Cahalan R. F.* Predictability in a Solvable Stochastic Climate Model// J. Atmos. Sci. — 1981. — **38.** — P. 504—513.
16. *Samelson R. M.* Stochastically Forced Current Fluctuations in Vertical Shear and over Topography// J. Geophys. Res. — 1989. — **94.** — P. 8207—8215.
17. *Saravanan R., McWilliams J. C.* Advective Ocean-atmosphere Interaction: An Analytical Stochastic Model with Implications for Decadal Variability//J. Climate. — 1998. — **11.** — P. 165—188.
18. *Sura P., Penland C.* Sensitivity of a Double-gyre Model to Details of Stochastic Forcing// Ocean Modelling. — 2002. — **4.** P. 327—345.
19. *Duan J., Lu K., Schmalfub B.* Invariant Manifolds for Stochastic Partial Differential Equations// The Annals of Probability. — 2003. — **31.** — P. 2109—2135.
20. *Duan J., Lu K., Schmalfub B.* Smooth Stable and Unstable Manifolds for Stochastic Evolutionary Equations// J. Dynamics and Diff. Eqns. — 2004. — **16**. — P. 949—972.
21. *Mohammed S.-E. A., Zhang T., Zhao H.* The Stable Manifold Theorem for Semilinear Stochastic Evolution Equations and Stochastic Partial Differential Equations// Memoirs of the American Mathematical Society. — 2008. — **196.** — P. 1—105.
22. *Wang W., Duan J.* A Dynamical Approximation for Stochastic Partial Differential Equations// J. Math. Phys. — 2007. — **48**, N 10. — P. 102—701.
23. *Wang W., Duan J.* Homogenized Dynamics of Stochastic Partial Differential Equations with Dynamical Boundary Conditions// Comm. Math.Phys. — 2007. — **275.** — P. 163—186.
24. *Wang W., Cao D., Duan J.* Effective Macroscopic Dynamics of Stochastic Partial Differential Equations in Perforated Domains// SIAM J. Math. Anal. — 2007. — **38.** — P. 1508—1527.
25. *Prato G. Da, Zabczyk J.* Stochastic Equations in Infinite Dimensions. — Cambridge University Press, 1992.
26. *Rozovskii B. L.* Stochastic Evolution Equations. — Boston: Kluwer Academic Publishers, 1990.
27. *Walsh J. B.* An Introduction to Stochastic Partial Differential Equations. Lecture Notes in Math. — 1986. — **1180.** — P. 265—439.
28. *Chow P. L.* Stochastic Partial Differential Equations. — New York: Chapman & Hall/CRC, 2007.
29. *Prevot C., Rokner M.* A Concise Course on Stochastic Partial Differential Equations. — Lecture Notes in Mathematics. — New York: Springer, 2007. — Vol. 1905.

30. *Oksendal B.* Stochastic Differential Equations. Sixth Ed. — New York: Springer-Verlag, 2003.
31. *Arnold L.* Stochastic DifferentiL Equations. — New York: John Wiley & Sons, 1974.
32. *Berger M. S.* Nonlinearity and Functional Analysis. — Academic Press, 1977.
33. *Zeidler E.* Applied Functional Analysis: Main Principles and Their Applications. — New York:Springer, 1995.
34. *Hunter J. K., Nachtergaele B.* Applied Analysis. — New Jersey: World Scientific, 2001.
35. *Zeidler E.* Applied Functional Analysis: Applications to Mathematical Physics. — New York: Springer, 1995.

Поступила 08.12.08